THE ELEMENTS OF MATHEMATICAL ANALYSIS,

ABRIDGED,

FOR THE USE OF STUDENTS.

WITH

NOTES,

DEMONSTRATIVE AND EXPLANATORY;

AND

A SYNOPSIS OF BOOK V. OF EUCLID.

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PREFACE.

IN this small Treatise on Mathematical Analysis, the most general Propositions ONLY are given; and in the Notes at the bottom of the pages, are pointed out the principal authors to be consulted by students. The whole to page 129, was printed in 1777; but owing to the Author's bad health, &c. it was not until 1783, that the same with the addition of the quarter sheet from page 129 to page 133 (which should have come in after Proposition VII.), was used here as a Text Book. As now published, the Definitions formerly omitted, are placed properly before the Axioms; and in the NOTES concluding the Analysis, from page 133 to page 148, are given demonstrations of several propositions, and corrections of several errata in the text; all other errata of consequence, being pointed out in the table following this preface.

Several Corollaries to Propositions XV. XVII. having been omitted by mistake in the printing, the reader is referred to the truly learned and judicious Mr Baron Maseres Dissertation on the Negative Sign, where the reduction of quadratic and cubic equations, &c. is given at great length, and in the most perspicuous and scientific manner.

The second Corollary (54) of Proposition XXI. being imperfect as PRINTED, may be corrected in manner following, viz. IN EVERY AFFECTED EQUATION, THE COEFFICIENTS OF THE INTERMEDIATE TERMS BETWEEN THE FIRST AND LAST, COMPREHEND ALWAYS divisors simple or compound of the last term, in the component parts of their sums and differences.

The rule of interest (115), as now given (207), is general for all rates PER CENT. PER ANNUM. And the rule of interest (210), is general also for all rates PER CENT. PER ANNUM, perfectly accurate, and more simple and easy in appplication, than any rule hitherto given.

The present Treatise is an abridgment of part of a comprehensive System of the Elements of Mathematical Analysis, common and fluxionary, now almost finished.

With regard to the Sign of Inequality, here introduced, the discriminating letters r, s, should have been in a smaller character, and placed above the angular point of the sign; or more SPACE should have been taken by the printer, to prevent any ambiguity when the symbols following are r, s, or any powers of the same.

J. R. J.

ERRATA.

ANALYSIS. Page 11, line 1, read $\left(=\frac{a}{b}\times\frac{c}{1}\right)$.

Page 16, Art. 17, lines 3, 4, read $ax-b=cx+d$, $ay^m+bc=d$.

—— 26, before Art. 30, add, SCHOLIUM III.—Lines 3, 4, dele *when m is some power of the number* 2.—Lines 5, 7, read binomial.

—— 27, *Note* *, line 1, read Willebrordus.

—— 28, Art. 32, line 3, read z^2 is wanting.—Art. 34, line 3, for $\sqrt{3q}$ read $\frac{2r}{3q}$.

—— 29, line 4, connect the cubic radicals with $+$.

—— 32, Art. 39, Ex. 2, 4, make first term x^3.

—— 34, Ex. 2, line 6, at the end, add, *in whole numbers*.

—— 42, Art. 48, lines 13, 14, insert last values of v, y, viz. 8, 12.

—— 44, Ex. 3, lines 1, 2, read $Q=-1$, $z^3+3z=36$, gives $2\,B=3$ (47).—Ex. 5, line 1, read 18252.—Ex. 6, line 3, read $v=8$.—Ex. 7, line 3, read $\frac{1}{2}\sqrt{-\frac{1}{3}}$.

—— 47, Art. 52, at the end, add, *with their signs changed*.

—— 49, Art. 56, line 6, before the colon, add, *taking in the last quotient*.—Art. 57, Ex. 2, line 3, read roots 3, 5, -6.—Ex. 3, line 1, read $106x$.

—— 50, Art. 58, Ex. 2, line 2, read 3, 5, -6; and line 3, read -3, -5, 6. —Ex. 3, lines 1, 3, read $106x$.—Art. 61, line 5, read $x+a=y$, or $x=y-a$.

—— 52, Art. 63, lines 7, 8, read $x^m\pm px^{m-1}$, &c. and $y\mp\frac{p}{m}=x$.

—— 75, line 1, prefix 85, Corol. V. and correct the Article Numbers to page 81.

—— 92, Case III. in values of n, v, for $2s+d$, read $2s+\frac{1}{4}d$.

—— 93. Case VII. in value of s, read $\frac{n-1}{2}$.—Case VIII. in values of a, n, read $-d\times\overline{2s-\frac{1}{4}d}$, and for $v=$&c. read $\frac{v\pm\sqrt{\&c.}}{d}+\frac{1}{2}$.

—— 111. Note ‡. See Abbot de Molieres in his Mathematical Lessons, London, 1730.

—— 118. Case VI. in value of r, dele $-$, and in value of s, to the fraction add $v-$. Case VIII. read $a=rv-s\times\overline{r-1}$; and hence correct the value of n.

—— 120, 121, lines 5, 16, read $na\times\frac{1}{r-v}$.—Art. 139, line 1, read *whole numbers*.

—— 123, Art. 141, lines 6, 7, for $2^{n\times1}$ read 2^{n+1}.

—— 131, Art. 157, line 1, read $Q:q=\frac{1}{V}:\frac{1}{v}$.

NOTES. Page 132, Art. 168, last line, dele $\times$ before $=$.

Page 136, line 20, for Hyp. read Since.

—— 138, Art. 187, line 2, read the roots; and line 3, read term.

SYNOPSIS. Page 4, line last but one, read ratio 5 : 7.

Page 13, Prop. D. Corol. line last, for greater less, read equally less.

—— 21, line 15, on the left, read Prop. X.

AXIOMS.

AXIOM I.

Every quantity is equal to itſelf.

AXIOM II.

The remainder of a quantity ſubtracted from itſelf, or from an equal quantity, is nothing.

AXIOM III.

Any whole is equal to all its parts.

AXIOM IV.

A quantity expreſſed one way, is equal to itſelf expreſſed any other way.

AXIOM V.

Quantities which are equal to one and the ſame quantity, or to equal quantities, are themſelves equal.

AXIOM VI.

Equimultiples of the ſame, or of equal quantities, are themſelves equal.

AXIOM VII.

If equal quantities be added to equal quantities, the ſums will be equal.

AXIOM VIII.

If equal quantities be ſubtracted from equal quantities, the remainders will be equal.

AXIOM IX.

If equal quantities be multiplied by equal quantities, the products will be equal.

AXIOM X.

If equal quantities be divided by equal quantities, the quotients will be equal.

AXIOM XI.

If equal quantities be equally involved or evolved, the powers or roots will be equal.

MATHEMATICAL ANALYSIS.

SECTION I. *The Algorithm of Quantity.*

1. $+$ *Plus.* The ſign of addition. $A+B$, ſum of A and B.

$-$ *Minus.* ——— ſubtraction. $A-B$, is B ſubtracted from A.

$\times$ *Multiplication.* $A\times B$ or AB, the product of A by B.

$\div$ *Diviſion.* $A\div B$, or $\frac{A}{B}$, the quotient of A divided by B.

$=$ *Equality.* Thus $A=B$, an equation.

: *Ratio.* Thus $A:B$, denotes the ratio of A to B.

$\overline{A+B}+\overline{C-D}$. The ſum of the compound quantities $A+B$ and $C-D$.

$\overline{A+B}-\overline{C-D}$. The difference.

$\overline{A+B}\times\overline{C-D}$. The product.

$\left.\begin{array}{r}\overline{A+B}\div\overline{C-D}\\ \text{Or, }\dfrac{A+B}{C-D}\end{array}\right\}$ The quotient.

Involution, or continual Multiplication. Raiſing of Powers.

$\left.\begin{array}{r}\text{Powers}\\ \text{of A.}\end{array}\right\}$ $A, AA, AAA, AAAA, AAAAA,$ *&c.*
Or,
$A, \quad A^2, \quad A^3, \quad A^4, \quad A^5,$ *&c.*

$\left.\begin{array}{r}\text{Powers of}\\ A\pm B.\end{array}\right\}$ $\overline{A\pm B}^{\,1}, \overline{A\pm B}^{\,2}, \overline{A\pm B}^{\,3}, \overline{A\pm B}^{\,4},$ *&c.*

Evolution,

Evolution, or continual division. Extraction of Roots.

Roots of A. $\left\{ \begin{array}{l} \sqrt{A},\ \sqrt[3]{A},\ \sqrt[4]{A},\ \sqrt[5]{A},\ \&c.\ \text{or,} \\ A^{\frac{1}{2}},\ A^{\frac{1}{3}},\ A^{\frac{1}{4}},\ A^{\frac{1}{5}},\ \&c. \end{array} \right.$

Roots of $A\pm B$. $\left\{ \begin{array}{l} \sqrt{\overline{A\pm B}},\ \sqrt[3]{\overline{A\pm B}},\ \sqrt[4]{\overline{A\pm B}},\ \&c. \\ \text{Or,} \\ \overline{A\pm B}|^{\frac{1}{2}},\ \overline{A\pm B}|^{\frac{1}{3}},\ \overline{A\pm B}|^{\frac{1}{4}},\ \&c. \end{array} \right.$

2. *The use of the* Vinculum, *in expressing the sums, differences, products, and quotients of compound quantities.*

Suppose $A=12$, $B=5$, $C=20$, and $D=15$,

$$\overline{A+B}+\overline{C-D}=\overline{12+5}+\overline{20-15}=17+5=22.$$

$$\overline{A-B}+\overline{C-D}=\overline{12-5}+\overline{20-15}=7+5=12.$$

$$\overline{C+D}-\overline{A+B}=\overline{20+15}-\overline{12+5}=35-17=18.$$

$$\overline{C+D}-A+B=\overline{20+15}-12+5=40-12=28.$$

$$\overline{C+D}-\overline{A-B}=\overline{20+15}-\overline{12-5}=35-7=28.$$

$$\overline{C+D}-A-B=\overline{20+15}-12-5=35-17=18.$$

$$\overline{A+B}\times C=\overline{12+5}\times 20=17\times 20=340.$$

$$A+B\times C=12+5\times 20=12+100=112.$$

$$\overline{A+B}\times\overline{C-D}=\overline{12+5}\times\overline{20-15}=17\times 5=85.$$

$$\overline{A+B}\times C-D=\overline{12+5}\times 20-15=340-15=325.$$

$$\overline{C+D}\div\overline{A-B}=\overline{20+15}\div\overline{12-5}=35\div 7=5.$$

Or, $\dfrac{C+D}{A-B}=\dfrac{20+15}{12-5}=\dfrac{35}{7}=5.$

$$\overline{C+D}\div\overline{B+A}=\overline{20+15}\div\overline{5+12}=\frac{35}{17}=2\frac{1}{17}.$$

$$\overline{C+D}\div B+A=\overline{20+15}\div 5+12=7+12=19.$$

Note. This sign ∽ denotes universally the difference of any two quantities.

Proposition I.

3. The sum of any number of quantities is *expressed*, by connecting quantities with their proper signs, and *uniting* such as can be *united*. But, in the addition of ratios, the product of the antecedents gives the antecedent of the sum, and that of the consequents, the consequent of the sum.

Ex. 1.

$$4ab + 2b^2 - 2abc - 5a^2b^2 + m + \frac{7a}{b} - \frac{4c^2}{a} + \frac{a^2}{c}$$

$$8ab - 2b^2 - 4abc + 3a^2b^2 - n + \frac{12a}{b} + \frac{3c^2}{a} + \frac{bx}{c}$$

$$12ab - 6abc - 2a^2b^2 + m - n + \frac{19a}{b} - \frac{c^2}{a} + \frac{a^2 + bx}{c}$$

Ex. 2.

$$8a^2b + \frac{10a^2}{b^2} + 12\sqrt{\frac{bc}{a}} - 3\sqrt{ab - x^2} + \sqrt[3]{a^5b} + \frac{a}{b}$$

$$3a^2b - \frac{4a^2}{b^2} - 7\sqrt{\frac{bc}{a}} + 12\sqrt{ab - x^2} - \sqrt[3]{a^5b} + \frac{c}{d}$$

$$11a^2b + \frac{6a^2}{b^2} + 5\sqrt{\frac{bc}{a}} + 9\sqrt{ab - x^2} + \frac{a}{b} + \frac{c}{d}$$

Ex 3.

$$\overline{A:B} + \overline{C:D} = AC:BD.$$

Ex. 4.

$$\overline{A:B} + \overline{B:C}\ (= AB:BC) = A:C.$$

Proposition II.

4. Subtraction in general takes place by connecting the quantities together, as in addition, supposing the signs of the subtrahend changed. But, in the subtraction of ratios, the terms of the subtrahend, or ablative ratio, must be inverted.

Ex. 1.

$$\begin{array}{l} 12ab - 6abc - 2a^2b^2 + m - n + \frac{19a}{b} - \frac{c^2}{a} + \frac{a^2}{c} + \frac{bx}{c} \\ 4ab - 2abc - 5a^2b^2 + m \qquad + \frac{7a}{b} - \frac{4c^2}{a} + \frac{a^2}{c} \\ \hline 8ab - 4abc + 3a^2b^2 - n + \frac{12a}{b} + \frac{3c^2}{a} \qquad + \frac{bx}{c} \end{array}$$

Ex. 2.

$$\begin{array}{l} 11a^2b + \frac{6a^2}{b^2} + 5\sqrt{\frac{bc}{a}} + 9\sqrt{ab - x^2} + \frac{a}{b} \qquad + \frac{c}{d} \\ 3a^2b - \frac{4a^2}{b^2} - 7\sqrt{\frac{bc}{a}} + 12\sqrt{ab - x^2} - \sqrt[3]{a^5b} + \frac{c}{d} \\ \hline 8a^2b + \frac{10a^2}{b^2} + 12\sqrt{\frac{bc}{a}} - 3\sqrt{ab - x^2} + \sqrt[3]{a^5b} + \frac{a}{b} \end{array}$$

Ex. 3.

$$\overline{A:B} - \overline{C:D}\ (= \overline{A:B} + \overline{D:C}) = AD:BC.$$

Ex. 4.

$$\overline{A:B} - \overline{B:C}\ (= \overline{A:B} + \overline{C:B}) = AC:B^2.$$

PROPOSITION III.

5. Multiplication connects the products of the factors with the signs + or — according as the factors have like or unlike signs; every term of the multiplier being drawn into each term of the multiplicand successively.

Ex. 1. $\overline{a+b} \times \overline{a+b} = a^2 + 2ab + b^2$. *Ex.* 2. $\overline{a+b} \times \overline{a-b} = a^2 - b^2$.

$$\begin{array}{r} a+b \\ a+b \\ \hline a^2+ab \quad\; \\ ab+b^2 \\ \hline a^2+2ab+b^2 \end{array} \qquad\qquad \begin{array}{r} a+b \\ a-b \\ \hline a^2+ab \quad\;\; \\ -ab-b^2 \\ \hline a^2-b^2. \end{array}$$

Ex.

Ex. 3. $\frac{a}{b} \times \frac{e}{d} = \frac{ae}{bd}$. Ex. 4. $\frac{a}{b} \times c \left(= \frac{a}{b} + \frac{c}{1}\right) = \frac{ac}{b}$.

Ex. 5. $\sqrt[m]{a} \times \sqrt[m]{b} = \sqrt[m]{ab}$.

Ex. 6. $\overline{3a - 2b + 2c} \times \overline{2a - 4b + 5c} =$
$6a^2 - 16ab + 19ac + 8b^2 - 18bc + 10c^2$

$$\begin{array}{l}
3a - 2b + 2c \\
2a - 4b + 5c \\
\hline
6a^2 - 4ab + 4ac \\
\quad - 12ab \qquad + 8b^2 - 8bc \\
\qquad\quad + 15ac \qquad - 10bc + 10c^2 \\
\hline
6a^2 - 16ab + 19ac + 8b^2 - 18bc + 10c^2.
\end{array}$$

Ex. 7.

Let m represent any number, then $m \times \overline{A : B} = A^m : B^m$.

6. Corollary. Powers of the same quantity may be multiplied by adding the exponents of the factors.

Thus: $a^m \times a^n = a^{m+n}$.

PROPOSITION IV.

7. In division, the quotient quantity is determined by comparing the dividend with the divisor, expunging all quantities that are common efficients in both, and making the result affirmative or negative, according as the signs of the divisor and dividend are like or unlike. Or, the quotient is universally that quantity, which, multiplied into the divisor, produces the dividend.

Ex. 1.

$a + b)\ a^2 + 2ab + b^2\ (a + b = \frac{a^2 + 2ab + b^2}{a + b}$

$$\begin{array}{l}
a^2 + ab \\
\hline
\quad ab + b^2 \\
\quad ab + b^2. \\
\hline
\quad\ \cdot \quad \cdot
\end{array}$$

Ex. 2. $a - b)\ a^2 - b^2\ (a + b = \frac{a^2 - b^2}{a - b}$

$$\begin{array}{l}
a^2 - ab \\
\hline
\quad ab - b^2 \\
\quad ab - b^2. \\
\hline
\quad\ \cdot \quad \cdot
\end{array}$$

Ex. 3. $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}$. *Ex.* 4. $\frac{a}{b} \div c = \frac{a}{bc}$.

Ex. 5. $\sqrt[m]{ab} \div \sqrt[m]{b} = \frac{\sqrt[m]{ab}}{\sqrt[m]{b}} \left(= \frac{\sqrt[m]{a} \times \sqrt[m]{b}}{\sqrt[m]{b}}\right) = \sqrt[m]{a}$.

Ex. 6.

$$
\begin{array}{l}
3a-2b+2c)\,6a^2-16ab+19ac+8b^2-18bc+10c^2\,(2a-4b+5c \\
\qquad\quad \underline{6a^2-\ \ 4ab+\ \ 4ac} \\
\qquad\qquad\quad -12ab+15ac+8b^2-18bc \\
\qquad\qquad\quad \underline{-12ab \qquad\quad +8b^2-\ \ 8bc} \\
\qquad\qquad\qquad\qquad 15ac \quad -10bc \quad +10c^2 \\
\qquad\qquad\qquad\qquad \underline{15ac \quad -10bc \quad +10c^2}
\end{array}
$$

Ex. 7. Let n represent any number, then $\overline{A:B} \div n = A^{\frac{1}{n}} : B^{\frac{1}{n}} = \sqrt[n]{A} : \sqrt[n]{B}$.

8. Corollary. Powers of the same quantity may be divided by subtracting the exponent of the divisor from that of the dividend. Thus: $a^m \div a^n = a^{m-n}$.

PROPOSITION V.

9. Involution in general takes place, by multiplying the exponent of the quantity to be involved by that of the power required. (5, 6.)

Thus: $\overline{a^m}|^n$ or a^m raised to the n^{th} power, is a^{mn}.

The involving of compound quantities will appear from the following table:

$$
\begin{array}{l}
\overline{a\pm b}|^1 = a \pm b \\
\overline{a\pm b}|^2 = a^2 \pm 2ab + b^2 \\
\overline{a\pm b}|^3 = a^3 \pm 3a^2b + 3ab^2 \pm b^3 \\
\overline{a\pm b}|^4 = a^4 \pm 4a^3b + 6a^2b^2 \pm 4ab^3 + b^4 \\
\overline{a\pm b}|^5 = a^5 \pm 5a^4b + 10a^3b^2 \pm 10a^2b^3 + 5ab^4 \pm b^5, \\
\quad \&c.\ \&c.
\end{array}
$$

10. Corollary. Hence, supposing m to represent the exponent of any power, then the terms of $\overline{a\pm b}|^m$, without their coefficients, will be as in the following series, *viz.*

a^m, $a^{m-1}b$, $a^{m-2}b^2$, $a^{m-3}b^3$, $a^{m-4}b^4$, &c.

And

And as for the coefficients, they will always be determined from this general product, *viz.*

$$1 \times m \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5}, \&c.$$

where, for the coefficient of any term, we must take the product of as many factors.

PROPOSITION VI.

11. Evolution in general takes place, by dividing the exponent of the quantity to be evolved by that of the root required, (7; 8).

Thus: $\sqrt[n]{a^m} = a^{\frac{m}{n}}$; and $\sqrt[n]{\overline{a \pm b}^m} = \overline{a \pm b}|^{\frac{m}{n}}$.

Hence, and from the last proposition, may be derived the following rule for the evolution of compound quantities:

Having found the first term of the root by this proposition, subtract its proper power from the quantity given to be evolved, and bring down the second term for a dividend; involve the root to the next inferior power to that given, and multiply this power by the exponent of the given power for a divisor; annex the quotient to the root, and involve the whole to the given power for a subtrahend; to the remainder bring down the term next to these affected by this subtraction for a new dividend; find the divisor as before, and proceed in this manner until all the terms of the quantity given to be evolved, are exhausted.

In applying this rule to numbers, let the numbers be distinguished into periods, by pointing according to the exponent of the power; and to obtain the decimal places in the root, to every remainder adjoin as many ciphers as the exponent contains units.

Square Root.

Ex. 1. $a^2 + 2ab + b^2\ (a + b$

a^2

$2 \times a = 2a)\ 2ab\ (b$

$\overline{a+b}^2 = a^2 + 2ab + b^2.$

Ex. 2. $a^2 + 2ab + b^2 - 2ac - 2bc + c^2\ (a + b - c$

a^2

$2 \times a = 2a)\ 2ab\ (b$

$\overline{a+b}^2 = a^2 + 2ab + b^2$

$2 \times a = 2a) - 2ac\ (-c$

$\overline{a+b-c}^2 = a^2 + 2ab + b^2 - 2ac - 2bc + c^2$

Cube

Cube Root.

Ex. 3. $a^3 - 3a^2b + 3ab^2 - b^3 \,(a - b$

a^3

$3 \times a^2 = 3a^2) - 3a^2b \,(-b$

$\overline{a-b}^3 = a^3 - 3a^2b + 3ab^2 - b^3$

Surfolid Root.

Ex. 4. $a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5 \,(a - b$

a^5

$5 \times a^4 = 5a^4) - 5a^4b \,(-b$

$a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5$

Ex. 5. $156.25 \,(12.5$ Square Root.

1

$1 \times 2 = 2) \, 5 \,(2$

$\overline{12}^2 = 144$

$12 \times 2 = 24) \, 122 \,(5$

$\overline{12.5}^2 = 156.25$

Ex. 6. $13312053 \,(237$ Cube Root.

8

$2^2 \times 3 = 12) \, 53 \,(3$

$\overline{23}^3 = 12167$

$\overline{23}^2 \times 3 = 1587) \, 11450 \,(7$

$\overline{237}^3 = 13312053$

Ex. 7. $4857532416 \,(264$ Fourth Root.

16

$2^3 \times 4 = 32) \, 325 \,(6$

456976

$\overline{26}^3 \times 4 = 70304) \, 287772 \,(4$

4857532416

12. Corollary. Since $\overline{a \pm b}^2 = a^2 \pm 2ab + b^2$, where $2ab$, the middle term, is twice the product of the square roots of the two extremes, it is evident that any quantity having these properties, will be a perfect square, and therefore the square roots of the extreme terms connected together with the sign of the middle term, will give the root required.

$x^2 \pm ax + \frac{1}{4}a^2$	Square Root.	$x \pm \frac{1}{2}a$
$100 + 40 + 4$		$10 + 2 = 12$
$100 + 10 + \frac{1}{4}$		$10 + \frac{1}{2} = 10\frac{1}{2}$
$49 + 7 + \frac{1}{4}$		$7 + \frac{1}{2} = 7\frac{1}{2}$
$36 + 3 + \frac{1}{16}$		$6 + \frac{1}{4} = 6\frac{1}{4}$
$16 + 6 + \frac{9}{16}$		$4 + \frac{3}{4} = 4\frac{3}{4}$

PROPOSITION VII.

13. When two ratios are equal, the four terms are said to be *proportional*, or to compose an ANALOGY: the two antecedent and consequent terms are called *homologous* quantities; and the terms of the same ratio, *analogous* quantities. And in any analogy, the product of the extreme terms is equal to that of the middle terms; and the analogous terms combined any how by addition and subtraction, and ordered alike, are always proportional or analogous.

Suppose the Ratio $A:B=C:D$; then $AD=BC$.

And,

The Ratio $\left\{\begin{array}{l} A+B:A=C+D:C \\ A+B:B=C+D:D \end{array}\right\}$ Composition.

The Ratio $\left\{\begin{array}{l} A \backsim B:A=C \backsim D:C \\ A \backsim B:B=C \backsim D:D \end{array}\right\}$ Division.

The Ratio $\left\{\begin{array}{l} A:A \underset{\backsim}{+} B=C:C \underset{\backsim}{+} D \\ B:A \underset{\backsim}{+} B=D:C \underset{\backsim}{+} D \end{array}\right\}$ Conversion.

The Ratio $A+B:A \backsim B=C+D:C \backsim D$ Mixtim.

14. Corol. 1. In any analogy the four terms may be taken reciprocally; and this interchanging of the antecedents and consequents, is called INVERSION: Also, when the terms are *all of the same kind*, the antecedents and consequents may be compared together; and this is called ALTERNATION or PERMUTATION.

Thus:

Suppose the Ratio $A:B=C:D$;

Then the Ratio, $A:C=B:D$, Alternation;

And the Ratio, $B:A=D:C$, Inversion.

Moreover, the terms of any one of the equal ratios, and the sums or differences of the *homologous* terms, are always analogous or proportional. Thus:

Suppose the Ratio $A:B=C:D$;

Then the Ratio, $A:B=A+C:B+D$, Syllepsis;

And the Ratio, $A:B=A \backsim C:B \backsim D$, Dialepsis.

15. Corol. 2. In any series of equal ratios, one antecedent has the same ratio to its consequent, as the sum of all the antecedents to the sum of all the consequents.

Suppose the Ratio $A:B=C:D=E:F=G:H$, &c.

Then the Ratio $A:B=A+C+E+G+$, &c.$:B+D+F+H+$, &c.

16. Corol.

16. Corol. 3. Since, when any four quantities are analogous or proportional, the product of the extreme terms is always equal to that of the middle terms (13), therefore any equation may be converted into an analogy, by taking the factors which compose one side, and making them the two extremes, and the factors which compose the other side of the equation, and making them the two means.

Thus: Suppose $A \times D = B \times C$;

Then, the Ratio $A : B = C : D$, or $D : C = B : A$.

The Ratio $A : C = B : D$, or $D : B = C : A$.

The Ratio $B : A = D : C$, or $C : D = A : B$.

Also, if it is $B = \sqrt{AC}$, or $B^2 = AC$; then

The Ratio $A : B = B : C$.

The Ratio $B : A = C : B$.

And the Ratio $C : B = B : A$.

For the *Algorithm* of quantity, see Sir Isaac Newton's Arithmetica Universalis, s' Gravesande's Algebra, Maclaurin's Algebra, and Simpson's Algebra. And for the doctrine of ratios, see Book V. of Euclid's Elements, and Saunderson's Algebra, or the Abridgement of the same.

SECTION II. *The Reduction of Equations.*

17. A simple equation is that which involves either one variable quantity, or a single power of one variable quantity combined any how with given quantities: As the equations $ax - b = cx + d$, and $ay^m + bc = d$. And those equations are called *compound*, which involve more than one variable quantity together with given quantities; such are these equations, $x + y = a$, and $x = by$.

18. An affected equation is that which involves different powers of the same variable quantity combined any how with given quantities, and is denominated according to the highest power of the variable qauntity. Thus, these equations are called *quadratics, viz.*

Case I. $x^2 + px = rg$. Case II. $x^2 - px = rg$. Case III. $px - x^2 = rg$.

The

The following are termed Cubics, *viz.*

$x^3 \pm p x^2 \pm q x = r$, $x^3 \pm p x^2 = r$, $x^3 \pm p x = r$, $p x - x^3 = r$, &c. &c.

The following Biquadratics, *viz.*

$x^4 \pm p x^3 \pm q x^2 \pm r x = s$, $x^4 \pm p x^3$, &c. $= s$, $p x$, &c. $- x^4 = s$, &c. &c.

And so on always according to the highest power of the variable quantity.

Proposition VIII.

19. Any term of an equation may be *transposed* from one side to the other, by changing its sign, (Ax. VII, VIII.)

Ex. 1. Suppose $4x = 30 + 3x$; then $(4x - 3x =)\ x = 30$.

Ex. 2. Suppose $x \pm b = a$; then $x = a \mp b$.

Ex. 3. Suppose $2x + b = a + x$; then $x = a - b$.

Proposition IX.

20. Any simple equation, that does not involve fractional quantities, may be reduced by transposing all the given qnantities to one side (19), and then dividing all the terms by the *coefficient* of the unknown quantity, (Ax. X.)

Ex. 1. Suppose $3x + 20 = 44$; then $3x = 24$ (19) and $x = \frac{24}{3} = 8$.

Ex. 2. Suppose $ax - b = cx + d$; then $(ax - cx =)\ \overline{a - c} \times x = b + d$ (19,) and $x = \frac{b + d}{a - c}$.

Ex. 3. Suppose $2ac + ab - ax = 3ac + 2ax - 5ab - dx$.

Ex. 4. Suppose $2ax + 5ab - 3bd = 2ab - 5ax + 7bd - ac - dx$.

Proposition X.

21. Any simple equation, involving fractional quantities, may be reduced, by multiplying all the terms by *each* denominator successively (Ax. IX.) Or, by bringing each side of the equation to

 a complete

a compleat fraction, reducing these fractions to a common denominator, and then casting off the common denominator, (Ax. IX.)

Ex. 1. Suppose $\frac{x}{5} + 4 = 10$; then multiplying all the terms by 5, we have $x + 20 = 50$, and therefore $x (= 50 - 20) = 30$ (19).

Or,

Since $\frac{x}{5} + 4 = 10$, therefore $\frac{x}{5} + \frac{20}{5} = \frac{50}{5}$, and $x = 30$, as before.

Ex. 2. Suppose $\frac{x}{5} + \frac{x}{3} = x - 7$; then multiplying all the terms by 5, we have $x + \frac{5x}{3} = 5x - 35$, and therefore all the terms of this equation being multiplied by 3, it is $3x + 5x = 15x - 105$, or, $7x = 105$ (19), and therefore $x = \frac{105}{7} = 15$ (20).

Or,

Since $\frac{x}{5} + \frac{x}{3} = x - 7$, therefore reducing all the terms to a common denominator, $\frac{3x + 5x}{15} = \frac{15x - 105}{15}$, which gives $x = 15$, as before.

Ex. 3. Suppose $\frac{2x}{3} + 4 = \frac{7x}{12} + 9$.

Ex. 4. Suppose $56 - \frac{3x}{4} = 48 - \frac{5x}{8}$.

Ex. 5. Suppose $\frac{ax}{x-b} = \frac{cx}{x-d}$.

Ex. 6. Suppose $\frac{cx}{2a} - \frac{ac}{2b} = x - \frac{4ad}{3c}$.

Ex. 7. Suppose $\frac{2ab^2x}{3c^2d} + \frac{5ac^2}{b^2} = \frac{6cd^2}{d^2} - 3x$.

PROPOSITION

PROPOSITION XI.

22. A ſimple equation, involving any power or root of an unknown quantity, combined any how with given quantities, may be reduced, by bringing the power or radical quantity to poſſeſs one ſide of the equation *alone* (19), and then evolving, or involving, according to the exponent of the power or radical quantity, (Ax. XI.)

Ex. 1. Suppoſe $x^2 = 144$; then $x = \sqrt{144} = 12$.

Ex. 2. Suppoſe $\frac{x^2 - 12}{3} = \frac{x^2 - 4}{4}$; then $x^2 = 36$, (21, 19), and therefore $x = \sqrt{36} = 6$.

Ex. 3. Suppoſe $\frac{5x^2}{16} - 8 = 12$; then $x = 8$.

Ex. 4. Suppoſe $ax^2 + bc = d$.

Ex 5. Suppoſe $ax^m - bc = cd + dx^m$.

Ex. 6. Suppoſe $\sqrt{4x + 16} = 12$; then by involution $4x + 16 = 144$ (8), and therefore $x = 32$ (19, 21).

Ex. 7. Suppoſe $\frac{\sqrt{5x}}{3} + 12 = 17$; then $x = 45$.

Ex. 8. Suppoſe $\sqrt{a + x} = b + \sqrt{x}$.

Ex. 9. Suppoſe $x + \sqrt{a^2 + x^2} = \frac{2a^2}{\sqrt{a^2 + x^2}}$.

Ex. 10. Suppoſe $\sqrt[3]{x^3 - a^3} = x - c$.

PROPOSITION XII.

23. Compound equations may be reduced by taking *values* of the *ſame* unknown quantity from different equations, and then equating theſe values, (Ax. V.) An equation will ariſe involving only one unknown quantity, ſuppoſing there were equations given for *each* unknown quantity.

Or,

Compound equations may be reduced by deriving values of the ſame unknown quantity from *ſome* of the equations, and ſubſtitut-

ing these values, instead of the quantities themselves, in all the *other* equations, (Ax. IV.)

Ex. 1.

$$\text{Suppose}\ \left\{\begin{array}{l} x+y=s \\ x-y=d \end{array}\right\} \ldots\ldots \begin{array}{l} x=s-y \\ x=d+y \end{array}\left\{ 2y=s-d.\right.$$

Thus: $y=\dfrac{s-d}{2}$, and therefore $x\ (=s-y=y+d)=\dfrac{s+d}{2}$.

Or,

$$\begin{array}{lll} x+y=s & \ldots\ldots & y=s-x \\ x-y=d & \ldots\ldots\ldots\ldots & x-\overline{s-x}=d \end{array}$$

Whence $x=\dfrac{s+d}{2}$, as before.

Ex. 2.

$$\text{Suppose}\ \left\{\begin{array}{l} x+y=a \\ x+z=b \\ y+z=c \end{array}\right\} \begin{array}{l} \ldots\ldots x=a-y \\ \ldots\ldots x=b-z \\ \ldots\ldots\ldots\ldots \end{array} \left\{\begin{array}{l} a-y=b-z \\ \\ y+z=c. \end{array}\right.$$

Thus: $z=\dfrac{c-a+b}{2}$.

And therefore $y\ (=a-b+z) = \dfrac{a-b+c}{2}$.

$$x\ (=a-y)=\frac{a-c+b}{2}.$$

Or,

$$\begin{array}{lll} x+y=a & \ldots\ldots & y=a-x \\ x+z=b & \ldots\ldots & z=b-x \\ y+z=c & \ldots\ldots\ldots\ldots & \overline{a-x}+\overline{b-x}=c \end{array}$$

Whence $2x=a-c+b$, as before.

Ex. 3.

$$\text{Suppose}\ \left\{\begin{array}{lll} x+y+z=a & \ldots\ldots\ldots & x=a-y-z \\ x+y+v=b & \ldots\ldots\ldots & x=b-y-v \\ x+z+v=c & \ldots\ldots\ldots & x=c-z-v. \\ y+z+v=d. & & \end{array}\right.$$

$$\begin{array}{lll} a-z=b-v & \ldots\ldots\ldots & z=a-b+v \\ a-y=c-v & & \\ y+z+v=d & \ldots\ldots\ldots & z=d-y-v \end{array}$$

$$\begin{array}{lll} v-b+a=d-y-v & \ldots & y=d-a+b-2v \\ a-y=c-v & \ldots\ldots\ldots & y=a-c+v \end{array}$$

$$v-c+a=d-a+b-2v.$$

Thus

Thus . . . . $v = \dfrac{b+c+d-2a}{3}$

Whence $y\ (=a-c+v) = \dfrac{a+b+d-2c}{3}$

$z\ (=a-b+v) = \dfrac{a+c+d-2b}{3}$

$x\ (=a-y-z) = \dfrac{a+b+c-2d}{3}$

Or,

$$\begin{array}{ll}
x+y+z=a \ \ldots\ldots & x=a-y-z \\
x+y+v=b & \\
x+z+v=c & \\
y+z+v=d & \\
\hline
a-z+v=b \ \ldots\ldots & z=a-b+v \\
a-y+v=c & \\
y+v+z=d & \\
\hline
a-y+v=c \ \ldots\ldots & y=a-c+v \\
y+a+2v-b=d & \\
\hline
\end{array}$$

$3v+2a-c-b=d$, as before.

Ex. 4. Suppose $\begin{cases} 4x-5y=2 \\ 6x-7y=4. \end{cases}$

Ex. 5. Suppose $\begin{cases} \dfrac{x}{2}+\dfrac{y}{3}=32 \\ \dfrac{x}{4}+\dfrac{y}{5}=18. \end{cases}$

Ex. 6. Suppose $\begin{cases} ax+by=m \\ cx+dy=n \end{cases}$

Ex. 7. Suppose $\begin{cases} ax+by+cz=m \\ dx+ey+fz=n \\ gx+hy+kz=p. \end{cases}$

Proposition XIII.

24. In the reduction of compound equations, if the quantity to be *exterminated* is of different dimensions in the different equations, then, from the equations in which it is found, others must

must be derived by comparing and equating the different values of its highest power, (Ax. V.) And when the highest power is not the same in two equations, one of the equations must be multiplied by the quantity to be *exterminated*, or some power of it, (Ax. IX.)

Ex. 1.

Suppose $\begin{cases} x^2 - y^2 = d \;\ldots\ldots\; x^2 = d + y^2 \\ x + y = s \;\ldots\ldots\; x^2 = \overline{s-y}|^2. \end{cases}$

Thus $y = \dfrac{s^2 - d}{2s} = \frac{1}{2}s - \frac{1}{2}\dfrac{d}{s}.$

And $x\,(= s - y) = \frac{1}{2}s + \frac{1}{2}\dfrac{d}{s}.$

Ex. 2.

Suppose $\begin{cases} x + y + \dfrac{y^2}{x} = a \;\ldots\; \overline{x + \dfrac{y^2}{x}}|^2 = \overline{a - y}|^2 \\ x^2 + y^2 + \dfrac{y^4}{x^2} = b \end{cases}$

Thus: $y = \dfrac{a^2 - b}{2a} = \frac{1}{2}a - \frac{1}{2}\dfrac{b}{a}.$

Ex. 3.

Suppose $\begin{cases} 2x + z = a \;\ldots\ldots\ldots\; 4x^2 = a^2 - 2az + z^2 \\ 2x^2 + 2y^2 + z^2 = b \\ x^2 - y^2 = z^2 \end{cases} \left. \begin{matrix} 3x^2 + y^2 = b \\ x^2 - y^2 = z^2 \end{matrix} \right\} 4x^2 = b + z^2.$

Thus: $z\left(= \dfrac{a^2 - b}{2a}\right) = \frac{1}{2}a - \frac{1}{2}\dfrac{b}{a}.$

Whence $x\,(= \frac{1}{2}a - \frac{1}{2}z) = \frac{1}{4}a + \frac{1}{4}\dfrac{b}{a}.$

And $y\,(= \sqrt{x^2 - z^2}\,) = \frac{1}{4}\sqrt{10b - 3a^2 - \dfrac{3b^2}{a^2}}.$

Proposition XIV.

25. All compound equations may be reduced by bringing the same unknown quantities, in different equations, to have the same

ſame coefficients by multiplication or diviſion; and then adding and ſubtracting theſe equations. This reduction may alſo ſometimes take place, by only multiplying or dividing ſome of the equations by a proper quantity, (Ax. VII, VIII, IX and X.)

Ex. 1. Suppoſe $\begin{cases} x + y = s \\ x - y = d \end{cases}$

Then by Addition, $2x = s + d$, and $x = \dfrac{s+d}{2}$

Alſo by Subtraction, $2y = s - d$, and $y = \dfrac{s-d}{2}$

Ex. 2. Suppoſe $\begin{cases} x + y + z = 26 \\ x - y \qquad = 4 \\ x - z \qquad = 6 \end{cases}$

Then by Addition, $3x = 36$, and $x = 12$

Whence $y = 8$, and $z = 6$.

Ex. 3. Suppoſe $\begin{cases} ax = \dfrac{b^2 x - b^3}{z} \\ x = \dfrac{az}{x - b} \end{cases}$

Then by Multiplication, $ax^2 = ab^2$, or $x = b$.

Ex. 4. Suppoſe $\left\{\begin{matrix} x + y = 12 \\ 5x + 3y = 50 \end{matrix}\right\}$ Then $\begin{cases} 5x + 5y = 60 \\ 5x + 3y = 50 \end{cases}$

Whence by Subtraction, $2y = 10$; thus: $y = 5$, and $x = 7$.

Ex. 5.

Suppoſe $\left\{\begin{matrix} 5x + 8y = 124 \\ 3x - 2y = 20 \end{matrix}\right\}$ Then $\left\{\begin{matrix} 15x + 24y = 372 \\ 15x - 10y = 100 \end{matrix}\right\}$ And $\begin{cases} 10x + 16y = 248 \\ 24x - 16y = 160 \end{cases}$

Whence by Subtraction, $34y = 272$; and by Addition, $34x = 408$: Thus: $y = 8$, and $x = 12$.

Ex. 6. Suppoſe $\begin{cases} 5x - 3y = 90 \\ 2x + 5y = 160. \end{cases}$

Ex.

Ex. 7. Suppose $\begin{cases} \dfrac{x}{2} - 12 = \dfrac{y}{4} + 8 \\ \dfrac{x+y}{5} + \dfrac{x}{3} - 8 = \dfrac{2y-x}{4} + 27. \end{cases}$

Ex. 8. Suppose $\begin{cases} \dfrac{x}{2} + \dfrac{y}{3} + \dfrac{z}{4} = 62 \\ \dfrac{x}{3} + \dfrac{y}{4} + \dfrac{z}{5} = 47 \\ \dfrac{x}{4} + \dfrac{y}{5} + \dfrac{z}{6} = 38. \end{cases}$

PROPOSITION XV.

26. Affected quadratic equations, (18) may be reduced, by adding, in the first and second cases, and subtracting in the third case, both sides of the equations to and from the square of half the coefficient of the term, involving the variable quantity itself, (12); and then extracting the square root on both sides. (Ax. XI.)

Case I. $x^2 + px = rq \ldots x = -\frac{1}{2}p + \sqrt{\frac{1}{4}p^2 + rq} = \dfrac{-p + \sqrt{p^2 + 4rq}}{2}$

Case II. $x^2 - px = rq \ldots x = \frac{1}{2}p + \sqrt{\frac{1}{4}p^2 + rq} = \dfrac{p + \sqrt{p^2 + 4rq}}{2}.$

Case III. $px - x^2 = rq \ldots x = \frac{1}{2}p \pm \sqrt{\frac{1}{4}p^2 - rq} = \dfrac{p \pm \sqrt{p^2 - 4rq}}{2}.$

Ex. 1. $x^2 + 6x = 40$ . . . . $x = 4.$
Ex. 2. $x^2 - x = 56$ . . . . $x = 8.$
Ex. 3. $20x - x^2 = 96$ . . . . . $x = 12$, and 8.
Ex. 4. $\frac{5}{2}x - x^2 = \frac{3}{2}$ . . . . $x = 1\frac{1}{2}$, and 1.
Ex. 5. Suppose $x^2 + \frac{5}{2}x = \frac{13}{2}.$
Ex. 6. Suppose $x^2 - \frac{1}{2}x = 22\frac{1}{2}.$
Ex. 7. Suppose $x^2 + \frac{1}{2}x = 39.$
Ex. 8. Suppose $x^2 + 10x = 20.$
Ex. 9. Suppose $x^2 + 6x = 11.$
Ex. 10. Suppose $x^2 + \frac{1}{11}x = \frac{107}{99}.$

SCHOLIUM

SCHOLIUM I.

27. The third case of affected quadratic equations, is said to be **AMBIGUOUS**, because the root (x), admits of *two different interpretations*, viz. one less than $\frac{1}{2}p$, and the other greater than $\frac{1}{2}p$, but less than p. From the equation, $px - x^2 = rq$, it is evident, that x is less than p; suppose therefore $x = \frac{1}{2}p$, then by substitution, $\frac{1}{4}p^2 = rq$, and in every other case $\frac{1}{4}p^2$, is *greater* than rq: Hence, then in completing the square, both sides of the equation must be subtracted from $\frac{1}{4}p^2$, which gives $\frac{1}{4}p^2 - px + x^2 = \frac{1}{4}p^2 - rq$, and therefore $\sqrt{\frac{1}{4}p^2 - px + x^2} = \sqrt{\frac{1}{4}p^2 - rq}$, (Ax. XI.); but the square root of $\frac{1}{4}p^2 - px + x^2$, may be $\frac{1}{2}p - x$, or, $x - \frac{1}{2}p$ (12), and thus: $x = \frac{1}{2}p \pm \sqrt{\frac{1}{4}p^2 - rq}$, according to the proposition.

28. Since the *greatest* magnitude of the quantity $\overline{p-x} \times x$ ($= px - x^2$) is $\frac{1}{4}p^2$, which arises by supposing $p - x = x$, or, $x = \frac{1}{2}p$: Therefore, as $\overline{p-x} \times x$ may represent any product whose increase is *limited*, the factors, $p - x$, and x, being parts of the same quantity p, it appears, that the greatest magnitude of any such product, may be determined, by *supposing the factors equal, and substituting in the product the value of the variable quantity derived from this equation.*

SCHOLIUM II.

29. Since the exponent of the square is always double that of the root (9), any equations falling under the following general forms, viz. $x^{2m} \pm px^m = rq$, and $px^m - x^{2m} = rq$, may be reduced to quadratics by substitution. Thus: Let $y = x^m$; then $y^2 = x^{2m}$ (9); and by Case I. and II. (26), in the equation $x^{2m} \pm px^m = rq$, we have

$$x = \sqrt[m]{\pm\frac{1}{2}p + \sqrt{\frac{1}{4}p^2 + rq}} = \sqrt[m]{\frac{\pm p + \sqrt{p^2 + 4rq}}{2}}.$$

Also, in the equation $px^m - x^{2m} = rq$, by Case III. (26), we have

$$x = \sqrt[m]{\frac{1}{2}p \pm \sqrt{\frac{1}{4}p^2 - rq}} = \sqrt[m]{\frac{p \pm \sqrt{p^2 - 4rq}}{2}},$$

Ex. 1. Suppose $27x^2 - x^4 = 176$; then $x = 4$, and $x = \sqrt{11}$.

Ex. 2. Suppose $x^6 + 4x^3 = 96$; then $x = 2$.

Ex. 3. Suppo [illegible] $x^8 - 1[illegible] x^4 = 5751$.

Ex. 4. Suppose $x^3 - 14x^{\frac{3}{2}} = 16$.

30. In resolving the equations $x^{2m} \pm px^m = \pm rq$, and $px^m - x^{2m} = \pm rq$, by reducing them to quadratics, the following rule, will, in many cases, be useful, [redacted] Let $R \pm \sqrt{S}$ represent any quadratic binomial, involving a radical quantity, *viz.* $\sqrt{S}$; suppose $\sqrt{R^2 - S} = Q$, and $\frac{Q + R}{2} = B^2$, then will $B \pm \sqrt{B^2 - Q}$ or $\sqrt{B^2 - Q} \pm B$ be the square root of the proposed binomial $R \pm \sqrt{S}$.

Ex. 1. Required the square root of $27 \pm \sqrt{704}$? Here $Q = 5$, $B = 4$, and therefore $4 \pm \sqrt{11}$ the root required.

Ex. 2. Required the square root of $-1 \pm \sqrt{-8}$? Here $Q = 3$, $B = 1$, and therefore $1 \pm \sqrt{-2}$ the root required.

Ex. 3. Required the square root of $3 \pm \sqrt{8}$?

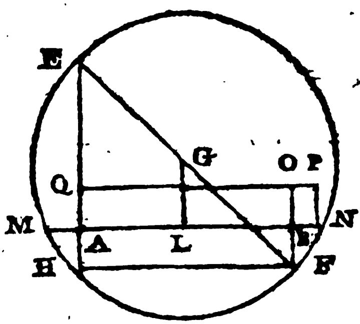
Ex. 4. Required the square root of $6 \pm \sqrt{20}$?

PROPOSITION XVI.

31. The root or roots of any affected quadratic equation, may be determined by means of a circle.

Case I. $x^2 + px = rq$.

CONSTRUCTION.



Draw the right line A B equal to p, and at right angles thereto, but on contrary sides, the two lines A E, B F, the former being equal to r, and the latter equal to q; join the points E F, and bisect E F in G; from the center G, with the radius G E, describe a circle meeting A E again in H, and cutting A B produced in M, N: Thus B N, or A M, will expound x.

Case

Case II. $x^2 - px = rq$.

CONSTRUCTION.

This case is constructed as the former; the value of x being MB, or AN.

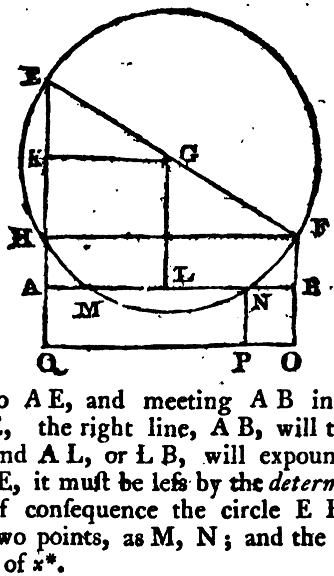
Case III. $px - x^2 = rq$.

DETERMINATION.

In this case rq is not greater than $\frac{1}{4}p^2$ (27.)

CONSTRUCTION.

Draw the right line AB equal to p, and perpendicular thereto, and on the same side the two lines AE, BF, the former being equal to r, and the latter equal to q; join the points E, F, and bisect EF in G; from the center G with radius GE, describe a circle cutting AE again in H; join the points H, F, and draw GK, parallel



to HF, as also GL parallel to AE, and meeting AB in L: Then, if GL be equal to GE, the right line, AB, will touch the circle in L, (16. e. 3.), and AL, or LB, will expound x: But if GL be not equal to GE, it must be less by the *determination*, (and 6. e. 2.), and of consequence the circle EHF, will intersect the line AB in two points, as M, N; and the lines AN, AM, will be the values of x*.

* These constructions were first given by Willebrodus Snellius, and the reader may consult Dr Simson's Notes upon book VI. of Euclid's Elements. And for a full illustration of all the propositions hitherto given for the reduction of equations, the reader may have recourse to Sir Isaac Newton's Arithmetica Universalis, Maclaurin's Algebra, Saunderson's Algebra, or the Abridgement of the same, Simpson's Algebra, Simpson's Select Exercises, and s' Gravesande's Algebra, &c.

PROPOSITION XVII.

32. In the general affected cubic equation, $z^3 - 3qz = 2r$, or $z^3 - 3qz - 2r = 0$, where the term involving the indeterminate quantity, z^2 is wanting; the value of z is *greater* than $\sqrt[3]{2r}$, as also *greater* than $\sqrt{3q}$, and may be expressed universally in manner following, *viz.*

$$z = \sqrt[3]{r + \sqrt{r^2 - q^3}} + \sqrt[3]{r - \sqrt{r^2 - q^3}}$$

Or,

$$z = \sqrt[3]{r + \sqrt{r^2 - q^3}} + \frac{q}{\sqrt[3]{r + \sqrt{r^2 - q^3}}}.$$

DEMONSTRATION.

Assume two quantities y, v, whereof y is the greater; and suppose these quantities to be such, that $y + v = z$, and $yv = q$: Then, by substituting in the proposed cubic equation, we have $2ry^3 - y^6 = q^3$, which gives $y^3 = r \pm \sqrt{r^2 - q^3}$ (29, 27), and of consequence $v^3 = r \mp \sqrt{r^2 - q^3}$; but since by supposition, y is greater than v, therefore $y^3 = r + \sqrt{r^2 - q^3}$, and $v^3 = r - \sqrt{r^2 - q^3}$, and hence

$$(y + v =)\, z = \sqrt[3]{r + \sqrt{r^2 - q^3}} + \sqrt[3]{r - \sqrt{r^2 - q^3}}$$

Or,

$$z = \sqrt[3]{r + \sqrt{r^2 - q^3}} + \frac{q}{\sqrt[3]{r + \sqrt{r^2 - q^3}}}.$$

Q. E. D.

33. Corol. 1. From the expression for z, it appears, that if r^2 should be less than q^3, the quantity $r^2 - q^3$, would be negative, (4) and its square root, therefore imaginary, (5): So that in this case the value of z cannot be directly determined by the general method of evolution, (11).

34. Corol. 2. If we change the sign of the term involving z, the equation becomes $z^3 + 3qz - 2r = 0$; where z is *less* than $\sqrt[3]{2r}$, as also *less* than $\sqrt{3q}$: Here then, the quantity $3q$ having a different sign from that in the equation of this proposition,

tion, therefore in the general expreſſion for z, the ſign of q, and of conſequence that of q^3, (5), muſt be changed; and hence in this caſe,

$$z = \sqrt[3]{r + \sqrt{r^2 + q^3}} - \sqrt[3]{r - \sqrt{r^2 + q^3}}.$$

Or,

$$z = \sqrt[3]{r + \sqrt{r^2 + q^3}} - \frac{q}{\sqrt[3]{r + \sqrt{r^2 + q^3}}}.$$

35. Corol. 3. If we change the ſign of $(2r)$, the abſolute term, the equation becomes $3qz - z^3 = 2r$, or $z^3 - 3qz + 2r = 0$; where if z be not equal to $\sqrt{q}$, it will have TWO REAL VALUES, one *leſs* than $\sqrt{q}$, and the other *greater* than $\sqrt{q}$, but *leſs* than $\sqrt{3q}$: Comparing this equation with that reſolved in this propoſition, it appears, that in the general expreſſion for z, (32), the ſign of the quantity r, muſt be changed, and hence in this caſe,

$$z = \sqrt[3]{-r + \sqrt{r^2 - q^3}} + \sqrt[3]{-r - \sqrt{r^2 - q^3}}.$$

Or,

$$z = \sqrt[3]{-r + \sqrt{r^2 - q^3}} + \frac{q}{\sqrt[3]{-r + \sqrt{r^2 - q^3}}}.$$

But in this equation, ſince the quantity r^2 is *leſs* than the quantity q^3, it is evident, that the value of z cannot be directly determined by the general method of evolution, (11).

The general reſolution of this equation, and of that in the firſt corollary, (33), will be given hereafter.

36. Corol. 4. By ſubſtituting $x - p$, inſtead of z in the equation, (32), $z^3 - 3qz - 2r = 0$, there will be produced the following cubic equation complete in all its terms, *viz.*

$$x^3 - 3px^2 + \overline{3p^2 - 3q} \times x - \overline{2r + p^3 - 3pq} = 0,$$

And from this general equation, the ſolution of any complete cubic equation, may be derived, by comparing the coefficients, and taking proper values of p, q, and r.

EXAMPLES OF CUBIC EQUATIONS.

(32.) *Ex.* 1. Suppoſe $z^3 - 36z = 91$. Here $q = 12$, $r = \frac{91}{2}$, and therefore $z = 4 + 3 = 7$.

Ex.

Ex. 2. Suppose $z^3 - 12z = 16$. Here $q = 4$, $r = 8$, and $r^2 - q^3 = 0$; whence $z = 2\sqrt[3]{r} = 4$.

Ex. 3. Suppose $z^3 - 96z = 576$.

Ex. 4. Suppose $z^3 - 2z = 21$.

(34.) *Ex.* 5. Suppose $z^3 + 6z = 20$. Here $q = 2$, $r = 10$, and therefore $z = 2$.

Ex 6. Suppose $z^3 + 30z = 117$. Here $q = 10$, $r = \frac{117}{2}$, and therefore $z = 5 - 2 = 3$.

Ex. 7. Suppose $z^3 + 9z = 30$.

Ex. 8. Suppose $z^3 + 24z = 58,7914$.

(36.) *Ex.* 9. Suppose $x^3 - 17x^2 + 54x - 350 = 0$. Here $p = \frac{17}{3}$, $3q = 42$, $2r = 407.92$, and the equation for z, is $z^3 - 42z = 407.92$, where $z = 9.287401$ nearly (32), and therefore $x\ (= z + p) = 14.954067$ nearly.

Ex. 10. Suppose $x^3 + 6x^2 - 183x - 2704 = 0$. Here $(-p = 2$, or$)$, $p = -2$, $3q = 105$, $2r = 2322$, and the equation for z, is $z^3 - 195z = 2322$, where $z = 18$, and therefore $x\ (= z + p) = 16$.

Ex. 11. Suppose $x^3 + 15x^2 + 84x - 100 = 0$. Here $p = -5$, $3q = -9$, $2r = 270$, and the equation for z, is $z^3 + 9z = 270$, where (34), $z = 6$, and therefore $x\ (= z + p) = 1$.

Ex. 12. Suppose $x^3 - 12x^2 + 40x - 512 = 0$.

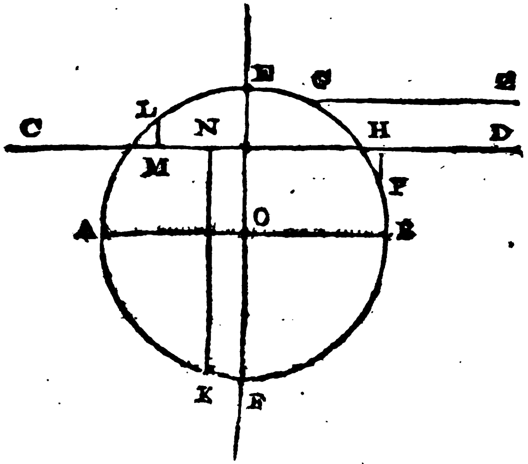
Ex. 13. Suppose $x^3 + 74x^2 + 8729x - 560783 = 0$.

Proposition XVIII.

37. *Admitting* affirmative and negative quantities, the three roots belonging to those cases of affected cubic equations, which could not

not be resolved by the last proposition, may be determined by means of a circle from the following

CONSTRUCTION.



General Equation. $\begin{cases} x^3 \pm 3px^2 \pm \overline{3p^2-3q}\times x + \overline{3pq-p^3-2r}=0 \\ x^3 \pm 3px^2 \pm \overline{3p^2-3q}\times x - \overline{2r+p^3-3pq}=0 \end{cases}$

With a radius A O equal to $2\sqrt{q}$, describe the circle A E B F, and draw the perpendicular diameters A B, E F; parallel to A B, and at the distance of p therefrom, draw C D above or below A B, according as the sign of the *second* term of equation is affirmative or negative; also parallel to A B, and at the distance of $\frac{2r}{q}$, therefrom draw Z G above or below A B, according as the sign of the *absolute* term is affirmative or negative; take the arch B P equal to one third of the arch B G, and from the point P divide the periphery of the circle into three equal parts P L, L K, and K P; from the points of division P, L, K, draw to C D the perpendicular lines P H, L M, K N, and these will be the three roots of the equation; the affirmative roots falling above, and the negative roots below C D.

38. Corol. 1. If $p=0$, then C D coincides with A B; and if $r=0$, then Z G coincides with A B.

39. Corol.

39. Corol 2. When $\frac{2r}{q}$ is greater than $2\sqrt{q}$, the construction fails.

Ex. 1. $x^3 - 12x^2 + 41x - 42 = 0$.
Here $p = 4$, $q = \frac{7}{3}$, and $r = 3$; whence $2\sqrt{q} = 3.054 = 3$ nearly, $\frac{2r}{q} = 2.571 = 2\frac{1}{2}$ nearly; and therefore $LM = 2$, $PH = 3$, and $NK = 7$.

Ex. 2. $x^3 + 9x^2 - 22x - 120 = 0$.

Here $p = 3$, $q = \frac{49}{3}$, and $r = 0$; whence $2\sqrt{q} = \frac{14}{\sqrt{3}} = 8.1$ nearly; and since, in this example, ZG coincides with AB (38), therefore $LM = 4$, $BH = -3$, and $MK = -10$.

Ex. 3. $x^3 - 17x^2 + 82x - 120 = 0$.

Here $p = \frac{17}{3} = 5\frac{2}{3}$, $q = \frac{43}{9}$, $2\sqrt{q} = 4.4$ nearly, and $\frac{2r}{q} = 4$ nearly; whence $KN = 3$, $PH = 4$, and $LM = 10$.

Ex. 4. $x^3 - 13x + 12 = 0$.

Here $p = 0$, $q = \frac{13}{3}$, $2\sqrt{q} = 4.2$ nearly, and $\frac{2r}{q} = 2.8$ nearly; thus, in this example, CD coincides with AB, and $PH = 1$, $LM = 3$, and $NK = -4$.

Ex. 5. $x^3 - \frac{7}{4}x + \frac{3}{4} = 0$.

Here $p = 0$, $q = \frac{7}{12}$, $2\sqrt{q} = 1.6$ nearly, and $\frac{2r}{q} = \frac{9}{7} = 1.3$ nearly; thus, in this example, CD coincides with AB, and $PH = \frac{1}{2}$, $LM = 1$, and $NK = -1\frac{1}{2}$.

Proposition XIX.

40. If one side of an equation is a fraction involving CONSTANT quantities *only*, and the other side a fraction involving ONE VARIABLE quantity in *each* of its terms, the *several* values of these variable quantities will be determined by taking all possible equimultiples of the correspondent terms of the given fraction.

Let

Let $m x = n y$ represent any equation involving two variable quantities; then, since $\frac{x}{y} = \frac{n}{m}$ (Ax. X.) $= \frac{2n}{2m} = \frac{3n}{3m} = \frac{4n}{4m}$, &c. $= \frac{\frac{1}{2}n}{\frac{1}{2}m} = \frac{\frac{1}{3}n}{\frac{1}{3}m} = \frac{\frac{1}{4}n}{\frac{1}{4}m}$, &c. (6), it is obvious, that, if

$x = n, 2n, 3n, 4n,$ &c. $\frac{1}{2}n, \frac{1}{3}n, \frac{1}{4}n,$ &c.

then $y = m, 2m, 3m, 4m,$ &c. $\frac{1}{2}m, \frac{1}{3}m, \frac{1}{4}m,$ &c.

(Ax. X.) Q. E. D.

41. Corol. 1. When an equation involves two variable quantities, it is said to be INDETERMINATE; because each of the variable quantities may be interpreted by several different numbers, and sometimes by the same numbers.

42. Corol. 2. From this proposition, it appears, that in the equation $m x = n y$, if x can be interpreted by n or any multiple of n, then may y be interpreted by m or an equimultiple of m. That is, in an indeterminate equation, involving two variable quantities, x, y, the values of x go on by n, the coefficient of y, and those of y by m, the coefficient of x: And therefore, having obtained the first values of x, y, the other values will be determined by a constant addition or subtraction of the coefficients of y, x, or of equimultiples of these quantities.

43. Corol. 3. Let A, B, C, represent three constant quantities; then, if $A x + B y = C$, it is evident, that, if x increases, y must decrease, and *vice versa*: But, if $A x - B y = C$, the two variable quantities, x, y, must either, both together, encrease, or both together decrease.

44. Corol. 4. From the equation $A x \pm B y = C$, we have $x = \frac{C \mp B y}{A}$ and $y = \frac{A x \backsim C}{B}$, (Ax. X.) And therefore, if by actual division, either of these quantities be reduced to more simple terms, then, by substituting for the fractional part of the quotient, a new equation will be obtained more simple than the original one.

EXAMPLES OF INDETERMINATE EQUATIONS.

Ex. 1. Suppose $9x - 7y = 6$.

Here then we have $\frac{3x - 2}{y} = \frac{7}{3}$, and therefore, by this proposition (40.)

E If

If $3x-2=7$, 14, 21, 28, &c.

Then $y=3$, 6, 9, 12, &c.

Whence, supposing x, y, to be whole numbers

$$\left.\begin{array}{r}3x-2=7\\ y=3\end{array}\right\}\begin{array}{l}x=3\\ y=3.\end{array}$$

Thus Corol. 2. (42.) and Corol. 3. (43.)

$x=3$, 10, 17, 24, 31, 38, &c. (Difference 7.)

$y=3$, 12, 21, 30, 39, 48, &c. (Do. 9.)

Ex. 2. Suppose $5x+9y=200$.

Here we have $\frac{40-x}{y}=\frac{9}{5}$, and therefore,

If $40-x=9$, 18, 27, 36,

Then $y=5$, 10, 15, 20.

Whence, since the first series terminates at 36, the quantities x, y, are limited to four values, in whole numbers.

$$\left.\begin{array}{r}40-x=9\\ y=5\end{array}\right\}\begin{array}{l}x=31\\ y=\ \ 5.\end{array}$$

Thus: $x=31$, 22, 13, 4, (Difference 9.)

$y=\ 5$, 10, 15, 20, (Do. 5.)

Ex. 3. Suppose $2x-y=100$.

Here we have $\frac{x}{100+y}=\frac{1}{2}$, and therefore,

If $x=1$ . . . . 51, 52, 53, &c.

Then $100+y=2$ . . . . . . 102, 104, 106, &c.

$$\text{Whence, supposing}\left.\begin{array}{r}x=\ 51\\ 100+y=102\end{array}\right\}\text{Then}\left\{\begin{array}{l}x=51\\ y=\ \ 2.\end{array}\right.$$

Thus: $x=51$, 52, 53, 54, 55, &c. (Diff. 1.)

$y=\ 2$, 4, 6, 8, 10, &c. (Do. 2.)

On

Or, by Corol. 4. (44.)

Since $2x - y = 100$, therefore $x = 50 + \frac{1}{2}y$; let $v = \frac{1}{2}y$, or $2v = y$; then, since $\frac{v}{y} = \frac{1}{2}$, therefore,

$$\begin{array}{lllllllll} v = & 1, & 2, & 3, & 4, & 5, & 6, & \&c. & (\text{Diff. } 1) \\ y = & 2, & 4, & 6, & 8, & 10, & 12, & \&c. & (\text{Do. } 2.) \\ (50 + \frac{1}{2}y =)\, x = & 51, & 52, & 53, & 54, & 55, & 56, & \&c. & (\text{Do. } 1.) \end{array}$$

Ex. 4. Suppose $14x - 19y = 11$.

Here, we have $\frac{14x - 11}{y} = \frac{19}{1}$, and therefore,

$$\begin{array}{llllllllllll} \text{If } 14x - 11 = & 19, & 38, & 57, & 76, & 95, & 114, & 133, & 152, & 171, & \&c. \\ \text{Then } \quad y = & 1, & 2, & 3, & 4, & 5, & 6, & 7, & 8, & 9, & \&c. \end{array}$$

Whence, supposing x, y, to be whole numbers,

$$\left.\begin{array}{r} 14x - 11 = 171 \\ y = 9 \end{array}\right\} \text{ Then } \left\{\begin{array}{l} x = 13 \\ y = 9. \end{array}\right.$$

$$\begin{array}{lllllll} \text{Thus: } x = & 13, & 32, & 51, & 70. & 89, & \&c. \quad (\text{Diff. } 19.) \\ \qquad\quad y = & 9, & 23, & 37, & 51, & 65, & \&c. \quad (\text{Do. } 14.) \end{array}$$

Or, by Corol. 4. (44.)

Since $14x = 19y + 11$, therefore, $x = \frac{19y + 11}{14} = y + \frac{5y + 11}{14}$; let $v = \frac{5y + 11}{14}$, then $y = \frac{14v - 11}{5} = 2 \times \overline{v - 1} + \frac{4v - 1}{5}$; let $z = \frac{4v - 1}{5}$, then $v = \frac{5z + 1}{4} = z + \frac{z + 1}{4}$; let $w = \frac{z + 1}{4}$, then $\frac{z + 1}{w} = \frac{4}{1}$, or $\frac{w}{z + 1} = \frac{1}{4}$; and therefore,

$$\left.\begin{array}{r} \text{If } w = 1 \\ z + 1 = 4 \end{array}\right\} \text{ Then } \left\{\begin{array}{l} w = 1 \\ z = 3 \end{array}\right\} \text{ And } v\ (= z + w) = 4.$$

Whence, $y\ (= 2 \times \overline{v - 1} + z) = 9$, and $x\ (= y + v) = 13$. And thus, the first values of x, y, may be brought to the most *simple determination possible.*

Ex. 5. Suppose $21x + 5y = 800$.

Ex. 6. Suppose $8y - 3x = 16$.

Ex. 7. Suppose $24x - 13y = 16$.

45. Corol. 5. By this proposition, we can determine the least whole numbers, that, divided by given numbers, shall have given remainders. Thus. Suppose we were to find the least whole number, that, divided by 3, 7, there shall remain 1, 3; let N represent the number sought, then $\frac{N-1}{3}$, $\frac{N-3}{7}$, are whole numbers; wherefore, suppose $\frac{N-1}{3} = P$, then $N = 3P + 1$, and $\frac{N-3}{7} = \frac{3P-2}{7}$; suppose $\frac{3P-2}{7} = Q$, then $P = \frac{7Q+2}{3} = 2Q + \frac{Q+2}{3}$; let $\frac{Q+2}{3} = R$, then $\frac{Q+2}{R} = \frac{3}{1}$, and therefore,

$$\left.\begin{matrix} Q+2=3 \\ R=1 \end{matrix}\right\} \text{ Then } \left\{\begin{matrix} Q=1 \\ R=1 \end{matrix}\right., \text{ and therefore } P=3.$$

Whence, $N = 10$ the least whole number required. And by continuing the series for $Q+2$, R, the other values of N may be obtained.

Or,

Since $P = 2Q + \frac{Q+2}{3}$; let $\frac{Q+2}{3} = 1$, or $Q = 1$, then $P = 3$, and $N = 10$, as before. And, by supposing $\frac{Q+2}{3} = 2, 3, 4,$ &c. the next greater values of N will be determined as before.

Again, Suppose we were to find the least whole number, that, divided by 28, 19, shall give 10, 12, for remainders. Let N represent the number required, then $\frac{N-10}{28}$, $\frac{N-12}{19}$, are whole numbers

numbers by supposition; wherefore suppose $\frac{N-10}{28}=P$, then $N=28\,P+10$, and therefore $\frac{N-12}{19}=\frac{28\,P-2}{19}=P+\frac{9\,P-2}{19}$; let $\frac{9\,P-2}{19}=Q$, then $P=\frac{19\,Q+2}{9}=2\,Q+\frac{Q+2}{9}$: Here, by taking $\frac{Q+2}{9}=1$, we have $Q=7$, and $N=430$, the least number required; and by taking $\frac{Q+2}{9}=2, 3$, &c. the next greater values of N will be obtained; or, the same may be determined, by forming the progressions, as in the last example.

46. The analysis given in this Corollary, may be extended to *any* number of divisors and remainders.

Ex. 1. To find a whole number, which, being divided by 3, 5, 7, 2, there shall remain 2, 4, 6, 0, respectively.

Let N represent the number required; then by supposition, $\frac{N-2}{3}$, $\frac{N-4}{5}$, $\frac{N-6}{7}$, and $\frac{N-0}{2}$ are whole numbers. Wherefore, suppose $\frac{N-2}{3}=P$; then $N=3\,P+2$, and therefore,

$$\frac{N-4}{5}=\frac{3\,P-2}{5}=\frac{5\,P-\overline{2\,P+2}}{5}=P-2\times\frac{P+1}{5}.$$

Suppose now $\frac{P+1}{5}=Q$; then $N=15\,Q-1$, and therefore $\frac{N-6}{7}=\frac{15\,Q-7}{7}=2Q-1+\frac{R}{7}$.

Suppose $\frac{R}{7}=S$; then $N=105\,S-1$, and therefore $\frac{N-0}{2}=52\,S+\frac{S-1}{2}$.

Let $\frac{S-1}{2}=0$; then $S=1$, and $N=104$, the *least* whole number

number required: And, by supposing $\frac{S-1}{2}=1, 2, 3,$ *&c.* the next succeeding values of N will be determined.

Ex. 2. To find a *whole* number, that, being divided by 16, 17, 18, 19, 20, there shall remain 6, 7, 8, 9, 10, respectively.

Let N represent the number required; then by supposition, $\frac{N-6}{16}$, $\frac{N-7}{17}$, $\frac{N-8}{18}$, $\frac{N-9}{19}$, and $\frac{N-10}{20}$, are whole numbers.

Suppose $\frac{N-6}{16}=P$; then $N=16P+6$, and therefore,

$$\frac{N-7}{17}=\frac{16P-1}{17}=\frac{17P-\overline{P+1}}{17}=P-\frac{P+1}{17}.$$

Suppose now $\frac{P+1}{17}=Q$; then $N=272Q-10$, and therefore $\frac{N-8}{18}=\frac{272Q-18}{18}=15Q-1+\frac{Q}{9}$.

Suppose now $\frac{Q}{9}=R$; then $N=2448R-10$, and therefore $\frac{N-9}{19}=\frac{2448R-19}{19}=128R-1+16\times\frac{R}{19}$.

Suppose $\frac{R}{19}=S$; then $N=46512S-10$, and therefore,

$$\frac{N-10}{20}=\frac{46512S-20}{20}=2325S-1+4\times\frac{S}{5}.$$

Let $\frac{S}{5}=1$; then $N=232550$, the *least* whole number required: And, by supposing $\frac{S}{5}=2, 3, 4,$ *&c.* the next succeeding values of N will be obtained *.

* See Dodson's Repository, v. 1. Simpson's Algebra and Select Exercises; as also Emerson's Algebra.

47. Corol.

47. Corol. 6. Any equation, involving two diffeaent powers of the same variable quantity, may be reduced by substitution to the form of an indeterminate equation, involving two variable quantities: And, *hence the resolution of all commensurate quadratic equations, as also of all commensurate cubic equations, wanting one term, biquadratic equations wanting two terms*, &c.

Examples of Quadratic Equations, (26, 27)

Table of Square Numbers.

Root	1,	2,	3,	4,	5,	6,	7,	8,	9,	0,	11,	12.
Square	1,	4,	9,	16,	25,	36,	49,	64,	81,	100,	121,	144.

Ex. 1. $x^2 + 6x = 40$. Here $\dfrac{40 - x^2}{x} = \dfrac{6}{1}$

$$\left.\begin{array}{l} 40 - x^2 = 6,\ 12,\ 18,\ 24 \\ \quad\ \ x = 1,\ 2,\ 3,\ 4 \end{array}\right\} \begin{array}{r} 40 - 24 = 16 \\ x = 4. \end{array}$$

being the least sq. No —

Ex. 2. $x^2 + 3x = 88$. Here $\dfrac{88 - x^2}{x} = \dfrac{3}{1}$.

$$\left.\begin{array}{l} 88 - x^2 = 3,\ 6,\ 9,\ 12,\ 15,\ 18,\ 21,\ 24 \\ \quad\ \ x = 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8 \end{array}\right\} \begin{array}{r} 88 - 24 = 64 \\ x = 8. \end{array}$$

Ex. 3. $x^2 - \frac{1}{2}x = 22\frac{1}{2}$. Here $\dfrac{x^2 - 22\frac{1}{2}}{x} = \dfrac{\frac{1}{2}}{1}$

$$\left.\begin{array}{l} x^2 - 22\tfrac{1}{2} = \tfrac{1}{2},\ 1,\ 1\tfrac{1}{2},\ 2,\ 2\tfrac{1}{2} \\ \quad\ \ x = 1,\ 2,\ 3,\ 4,\ 5 \end{array}\right\} \begin{array}{r} 2\tfrac{1}{2} + 22\tfrac{1}{2} = 25 \\ x = 5. \end{array}$$

Ex. 4 $5x - x^2 = 6$. Here $\dfrac{x^2 + 6}{x} = \dfrac{5}{1}$

$$\left.\begin{array}{l} x^2 + 6 = 5,\ 10 \\ \quad\ x = 1,\ 2 \end{array}\right\} \begin{array}{r} 10 - 6 = 4 \\ x = 2 \end{array} \left\{\begin{array}{c} 15 \\ 3 \end{array}\right\} \begin{array}{r} 15 - 6 = 9 \\ x = 3. \end{array}$$

Ex. 5.

Ex. 5. $x^6 + 4x^3 = 96$. Let $v = x^3$, then $v^2 + 4v = 96$ (27), and $\frac{96 - v^2}{v} = \frac{4}{1}$

$$\left.\begin{array}{l} 96 - v^2 = 4,\ 8,\ 12,\ 16,\ 20,\ 24,\ 28,\ 32 \\ v = 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8 \end{array}\right\} \begin{array}{r} 96 - 32 = 64 \\ v = 8. \end{array}$$

And $x = \sqrt[3]{v} = 2$.

Or,

Since $v^2 + 4v = 96$, therefore $\frac{v^2}{24 - v} = \frac{4}{1}$

$$\left.\begin{array}{l} v^2 = 4,\ 16,\ 36,\ 64 \\ 24 - v = 1,\ 4,\ 9,\ 16 \end{array}\right\} v = 8 \text{ as before.}$$

Examples of Deficient Cubic Equations.
(32, 33, 34, 35.)

Table of Cube Numbers.

Root	1,	2,	3,	4,	5,	6,	7,	8,	9,	10,	11,	12.
Cube	1,	8,	27,	64,	125,	216,	343,	512,	729,	1000,	1331,	1728.

Ex. 1. $z^3 - 6z = 40$. Here $\frac{z^3 - 40}{z} = \frac{6}{1}$

$$\left.\begin{array}{l} z^3 - 40 = 6,\ 12,\ 18,\ 24 \\ z = 1,\ 2,\ 3,\ 4 \end{array}\right\} \begin{array}{r} 24 + 40 = 64 \\ z = 4. \end{array}$$

Ex. 2. $z^3 + 84z = 400$. Here $\frac{400 - z^3}{z} = \frac{84}{1}$

$$\left.\begin{array}{l} 400 - z^3 = 84,\ 168,\ 252,\ 336 \\ z = 1,\ 2,\ 3,\ 4 \end{array}\right\} \begin{array}{r} 400 - 336 = 64 \\ z = 4. \end{array}$$

Ex. 3.

Ex. 3. $z^3+12z=32$. Here $\dfrac{32-z^3}{z}=\dfrac{12}{1}$

$$\left.\begin{array}{r} 32-z^3=12,\ 24 \\ z=\ \ 1,\ \ 2 \end{array}\right\}\ \begin{array}{r} 32-24=8 \\ z=2. \end{array}$$

(33.) *Ex.* 4. $z^3-7z=6$. Here $\dfrac{z^3-6}{z}=\dfrac{7}{1}$

$$\left.\begin{array}{r} z^3-6=7,\ 14,\ 21 \\ z=1,\ \ 2,\ \ 3 \end{array}\right\}\ \begin{array}{r} 21+6=27 \\ z=\ 3. \end{array}$$

(33.) *Ex.* 5. $z^3-13z=12$. Here $\dfrac{z^3-12}{z}=\dfrac{13}{1}$

$$\left.\begin{array}{r} z^3-12=13,\ 26,\ 39,\ 52 \\ z=\ 1,\ \ 2,\ \ 3,\ \ 4 \end{array}\right\}\ \begin{array}{r} 52+12=64 \\ z=\ 4. \end{array}$$

(35.) *Ex.* 6. $21z-z^3=20$. Here $\dfrac{z^3+20}{z}=\dfrac{21}{1}$

$$\left.\begin{array}{r} z^3+20=21 \\ z=\ 1 \end{array}\right\}\ \begin{array}{r} 21-20=1 \\ z=1 \end{array}\ \left\{\begin{array}{r} 42,\ 63,\ 84 \\ 2,\ \ 3,\ \ 4 \end{array}\right\}\ \begin{array}{r} 84-20=64 \\ z=\ 4. \end{array}$$

(35.) *Ex.* 7. $39z-z^3=70$. Here $\dfrac{z^3+70}{z}=\dfrac{39}{1}$

$$\left.\begin{array}{r} z^3+70=39,\ 78 \\ z=\ 1,\ \ 2 \end{array}\right\}\ \begin{array}{r} 78-70=8 \\ z=2 \end{array}\left\{\begin{array}{r} 117,\ 156,\ 195 \\ 3,\ \ 4,\ \ 5 \end{array}\right\}\ \begin{array}{r} 195-70=125 \\ z=\ 5. \end{array}$$

(35.) *Ex.* 8. $\frac{7}{4}z-z^3=\frac{3}{4}$. Here $\dfrac{z^3+\frac{3}{4}}{z}=\dfrac{\frac{7}{4}}{1}$

$$\left.\begin{array}{r} z^3+\frac{3}{4}=\frac{7}{4} \\ z=1 \end{array}\right\}\ \begin{array}{r} \frac{7}{4}-\frac{3}{4}=1 \\ z=1 \end{array}\ \left\{\begin{array}{c} \frac{7}{8} \\ \frac{1}{2} \end{array}\right\}\ \begin{array}{r} \frac{7}{8}-\frac{3}{4}=\frac{1}{8} \\ z=\frac{1}{2}. \end{array}$$

Ex. 9. $9x^2-x^3=100$. Here $\dfrac{100+x^3}{x^2}=\dfrac{9}{1}$ Or, $\dfrac{x^2}{100+x^3}=\dfrac{1}{9}$

$$\left.\begin{array}{r} x^2=1,\ \ 4,\ \ 9,\ \ 16,\ \ 25 \\ 100+x^3=9,\ 36,\ 81,\ 144,\ 225 \end{array}\right\}\begin{array}{l} 25=5^2=x^2 \\ 225-100=125=5^3=x^3 \end{array}$$

 The

Thus: we see in how very easy a manner these cubic equations may be resolved, which we formerly found could not be reduced by the general method of evolution, (33, 35.) And, we proceed in the same manner, in resolving deficient equations of all degrees.

48. Corol. 7. When an equation is proposed, involving more than two variable quantities, or more than two different powers of the same variable quantity; by substituting for the sum, difference, product, quotient, &c. of two or more of these quantities, the equation may be transformed into another, involving only two variable quantities, or two different powers of the same variable quantity.

Example.

Suppose $6x + 7y + 8z = 100$; to find all the values of x, y, z, in whole numbers.

Let $(6x + 8z =)\ 2 \times \overline{3x + 4z} = 2v$; then $2v + 7y = 100$; and $\frac{50 - v}{y} = \frac{7}{2}$

$$\left.\begin{aligned} 50 - v &= 7 \\ y &= 2 \end{aligned}\right\} \begin{aligned} v &= 43,\ 36,\ 29,\ 22,\ 15. \quad \text{(Dif. 7.)} \\ y &= 2,\ \ 4,\ \ 6,\ \ 8,\ 10. \quad \text{(Do. 2.)} \end{aligned}$$

Taking $v = 43$; we have $3x + 4z = 43$, or $\frac{x}{43 - 4z} = \frac{1}{3}$ and hence,

$$\left.\begin{aligned} x &= 1 \\ 43 - 4z &= 43 \end{aligned}\right\} \begin{aligned} x &= 1,\ 5,\ 9,\ 13. \quad \text{(Dif. 4.)} \\ z &= 10,\ 7,\ 4,\ 1. \quad \text{(Do. 3.)} \end{aligned}$$

Thus: when $y = 2\ \ 2\ \ 2\ \ 2$

$x = 1,\ 5,\ 9,\ 13$ $= 4$

$z = 10,\ 7,\ 4,\ 1.$

Taking $v = 36$ . . . $3x + 4z = 36$ . . . . $3x = \overline{9 - z}$ $\times 4$ . . . $\frac{x}{9 - z} = \frac{4}{3}.$

$$\left.\begin{aligned} x &= 4 \\ 9 - z &= 3 \end{aligned}\right\} \begin{aligned} x &= 4,\ 8. \quad \text{(Dif. 4.)} \\ z &= 6.\ 3. \quad \text{(Do. 3.)} \end{aligned}$$

Thus: when $y = 4\ \ 4$

$x = 4,\ 8$ $= 2$

$z = 6,\ 3.$

Taking

Taking $v=29$ . . . $3x+4z=29$ . . . . $\frac{x}{29-4z}=\frac{1}{3}$

$$\left.\begin{array}{r} x=1,\ 2,\ 3 \\ 29-4z=3,\ 6,\ 9 \end{array}\right\} \begin{array}{ll} x=3,\ 7. & \text{(Dif. 4.)} \\ z=5,\ 2. & \text{(Do. 3.)} \end{array}$$

Thus: when $y=6$ 6

$x=3,\ 7$ = 2

$z=5,\ 2.$

Taking $v=22$ . . . $3x+4z=22$ . . . $\frac{x}{11-2z}=\frac{2}{3}$

$$\left.\begin{array}{r} x=2 \\ 11-2z=3 \end{array}\right\} \begin{array}{ll} x=2,\ 6 & \text{(Dif. 4.)} \\ z=4,\ 1 & \text{(Do. 3.)} \end{array}$$

Thus: when $y=8$ 8

$x=2,\ 6$ = 2.

$z=4,\ 1.$

Taking $v=15$ . . . $3x+4z=15$ . . . $\frac{5-x}{z}=\frac{4}{3}$ . . .

$$\left.\begin{array}{r} 5-x=4 \\ z=3 \end{array}\right\} \begin{array}{l} x=1 \\ z=3. \end{array}$$

Thus: when $y=10$

$x=1$ = 1 11 Ans^{rs}

$z=3$

And these are all the whole numbers by which x, y, z, can be expounded *.

Section III. Of the Composition and Resolution of Quantity.

Proposition XX.

49. The root of *any cubic binomial, involving a quadratic surd quantity*, may be determined by the resolution of a COMMENSURATE cubic equation, wanting the second term.

* See Dodson's Repository, v. 2. Simpson's Select Exercises and Algebra.

Let $R \pm \sqrt{S}$ represent any cubic binomial, involving the radical quantity $\sqrt{S}$; suppose $\sqrt[3]{R^2 - S} = Q$, and let $2B$ be the value of z in this *commensurate* cubic equation, $z^3 - 3Qz = 2R$ (47); then will $B \pm \sqrt{B^2 - Q}$ be the cube root of the proposed binomial $R \pm \sqrt{S}$.

Ex. 1. Suppose $R = 20$, and $S = 392$; then $Q = 2$, and the equation $z^3 - 6z = 40$, gives $2B = 4$ (47), whence $B \pm \sqrt{B^2 - Q} = 2 \pm \sqrt{2}$.

Ex. 2. Suppose $R = 25$, and $S = 968$; then $Q = -7$, and the equation $z^3 + 21z = 50$, gives $2B = 2$ (47), whence, $B \pm \sqrt{B^2 - Q} = 1 \pm \sqrt{8}$.

Ex. 3. Suppose $R = 18$, and $S = 325$; then $Q = -2$, and the equation $z^3 + 12z = 288$, gives $2B = 6$ (47), whence $B \pm \sqrt{B^2 - Q} = \frac{3}{2} \pm \sqrt{3\frac{1}{4}}$.

Ex. 4. Suppose $R = 10$, and $S = 108$; then $Q = -2$, and the equation $z^3 + 6z = 20$, gives $2B = 2$ (47), whence, $B \pm \sqrt{B^2 - Q} = 1 \pm \sqrt{3}$.

Ex. 5. Suppose $R = 135$, and $S = 18152$; then $Q = -3$, and the equation, $z^3 + 9z = 270$, gives $2B = 6$ (47), whence $B \pm \sqrt{B^2 - Q} = 3 \pm \sqrt{12}$.

Ex. 6. Suppose $R = 68$, and $S = 4374$; then $Q = 5\sqrt[3]{2}$, and the equation, $z^3 - 15\sqrt[3]{2} \times z = 136$, by supposing $v = z \times \sqrt[3]{2}$, becomes $v^3 - 30v = 272$, which gives $v = 8$ (47); and therefore, $B \pm \sqrt{B^2 - Q} = \dfrac{4 - \sqrt{6}}{\sqrt[3]{2}}$.

Ex. 7. Suppose $R = 3$, and $S = -\dfrac{100}{27}$; then $Q = \dfrac{7}{3}$, and the equation, $z^3 - 7z = 6$, gives $2B = 3$ (47), whence $B \pm \sqrt{B^2 - Q} = \frac{3}{2} \pm \frac{1}{2}\sqrt{-\frac{1}{3}}$.

Ex. 8. Suppose $R = -10$, and $S = -243$; then $Q = 7$, and the equation, $21z - z^3 = 20$, gives $2B = 1$, as also $2B = 4$ (47); whence, taking $2B = 1$, we have $B \pm \sqrt{B^2 - Q} = \frac{1}{2} \pm \frac{3}{2}\sqrt{-3}$, where $\frac{1}{2} + \frac{1}{2}\sqrt{-3}$, is the root of $-10 - \sqrt{-243}$, and

and $\frac{1}{2}-\frac{1}{2}\sqrt{-3}$, that of $-10+\sqrt{-243}$; and taking $2B=4$, we have $B\pm\sqrt{B^2-Q}=2\pm\sqrt{-3}$.

Scholium.

50. As, by this proposition, the root of *any* cubic binomial, may be determined; so thereby are we enabled to bring to a *direct solution*, all affected cubic equations whatsoever, (32, 33, 34, 35, 36.) Let $z^3-3qz-2r=0$, represent any affected cubic equation wanting the second term, and let $m\pm\sqrt{n}$ represent the cube root of the binomial $r\pm\sqrt{r^2-q^3}$; then, since $z=2m$ (32), or $z-2m=0$, if we divide $z^3-3qz-2r$ by $z-2m$, we shall obtain this quadratic equation $z^2+2mz+\overline{4m^2-3q}=0$, which being reduced, (26), gives $z=-m\pm\sqrt{3q-3m^2}$ *admitting* affirmative and negative quantities: But ($y\times v=\overline{m+\sqrt{n}}\times\overline{m-\sqrt{n}}=$) $m^2-n=q$ (32), and therefore, by substitution, $z=-m\pm\sqrt{-3n}$. Thus: *admitting* affirmative and negative quantities; in the cubic equation, $z^3-3qz-2r=0$, we have

$$1.\ z=2m$$

$$2.\ z=-m+\sqrt{-3n}$$

$$3.\ z=-m-\sqrt{-3n}.$$

And therefore, in the general complete cubic equation (36), $x^3-3px^2+\overline{3p^2-3q}\times x-\overline{2r+p^3-3pq}=0$; since $x=p+z$, therefore,

$$1.\ x=p+2m$$

$$2.\ x=p-m+\sqrt{-3n}$$

$$3.\ x=p-m-\sqrt{-3n}.$$

51. In the application of this general equation to practise, the following observations may be attended to, (1.) That the third and fourth terms may be represented either as affirmative or negative, (37), by properly arranging these terms, (2.) That, if $p=0$, then

$$1.\ x\ (=z)=2m$$

$$2.\ x\ (=z)=-m+\sqrt{-3n}$$

$$3.\ x\ (=z)=-m-\sqrt{-3n}.$$

(3.) That

(3.) That, if $q = 0$, then $m = \sqrt{n}$, (32); whence,

$$1.\ z = 2m\ (= \sqrt[3]{2r})$$

$$2.\ z = -m + \sqrt{-3m^2}$$

$$3.\ z = -m - \sqrt{-3m^2};$$

And therefore,

$$1.\ x = p + 2m$$

$$2.\ x = p - m + \sqrt{-3m^2}$$

$$3.\ x = p - m - \sqrt{-3m^2}.$$

(4.) That, if $r = 0$, then $m = 0$, and $q = -n$; whence,

$$1.\ z = 0$$

$$2.\ z = \sqrt{3q}$$

$$3.\ z = -\sqrt{3q}.$$

And therefore,

$$1.\ x = p$$

$$2.\ x = p + \sqrt{3q}$$

$$3.\ x = p - \sqrt{3q}.$$

Ex. 1. $x^3 - 12x^2 + 41x - 42 = 0.$

Here $p = 4$, $q = \frac{7}{3}$, $r = 3$, and $r \pm \sqrt{r^2 - q^3} = 3 \pm \sqrt{-\frac{100}{27}}$; whence $m = \sqrt[3]{n} = \frac{3}{2} \pm \frac{1}{2}\sqrt{-\frac{1}{3}}$ (49), and therefore,

$$1.\ x\ (= p + 2m = 4 + 3) = 7$$

$$2.\ x\ (= p - m + \sqrt{-3n} = \tfrac{5}{2} + \tfrac{1}{2}) = 3$$

$$3.\ x\ (= p - m - \sqrt{-3n} = \tfrac{5}{2} - \tfrac{1}{2}) = 2.$$

Ex. 2. $x^3 + 15x^2 + 84x - 100 = 0.$

Here $p = -5$, $q = -3$, $r = 135$, and $r \pm \sqrt{r^2 - q^3} = 135 \pm \sqrt{18252}$; whence, $m = \sqrt[3]{n} = 3 \pm \sqrt{12}$ (49), and therefore,

$$1.\ x\ (= p + 2m) = -5 + 6 = 1$$

$$2.\ x\ (= p - m + \sqrt{-3n}) = -8 + \sqrt{-36}$$

$$3.\ x\ (= p - m - \sqrt{-3n}) = -8 - \sqrt{-36}.$$

Ex. 3. $x^3 + 9x^2 - 22x - 120 = 0.$

Here $p = -3$, $q = \frac{49}{3}$, and $r = 0$; whence,

$$1.\ x\ (= p) = -3$$

$$2.\ x\ (= p + \sqrt{3q}) = -3 + 7 = 4$$

$$3.\ x\ (= p - \sqrt{3q}) = -10.$$

Ex.

Ex. 4. $x^3 - 9x^2 + 27x - 35 = 0$.

Here $p = 3$, $q = 0$, $r = 4$; whence, $2m = \sqrt[3]{2r} = 2$, and therefore,

$$1.\ x\ (= p + 2m) = 5$$
$$2.\ x\ (= p - m + \sqrt{-3m^2}) = 2 + \sqrt{-3}$$
$$3.\ x\ (= p - m - \sqrt{-3m^2}) = 2 - \sqrt{-3}.$$

Ex. 5. $x^3 - 6x^2 - 9x + 54 = 0$.

Here $p = 2$, $q = 7$, $r = -10$, and $r \pm \sqrt{r^2 - q^3} = -10 \pm \sqrt{-243}$; whence, $m \pm \sqrt{n} = \frac{1}{2} \pm \frac{1}{2}\sqrt{-3}$, as also $m \pm \sqrt{n} = 2 \pm \sqrt{-3}$ (49), and therefore, taking the first value of $m \pm \sqrt{n}$

$$1.\ x\ (= p + 2m) = 3$$
$$2.\ x\ (= p - m + \sqrt{-3n}) = 6$$
$$3.\ x\ (= p - m - \sqrt{-3n}) = -3$$

Or, taking the second value of $m \pm \sqrt{n}$

$$1.\ x\ (= p + 2m) = 6$$
$$2.\ x\ (= p - m + \sqrt{-3n}) = 3$$
$$3.\ x\ (= p - m - \sqrt{-3n}) = -3.$$

PROPOSITION XXI.

52. *Admitting* affirmative and negative quantities; and supposing affected equations of *all degrees* to be produced by a multiplication of binomial factors. Every affected equation will have as many roots as there are units in the exponent of the highest power of the variable quantity; and, if the terms of the equation are alternately affirmative and negative, the roots will be all affirmative; but, if the terms are all affirmative, the roots will be all negative: Moreover, the coefficient of the second term, will be the sum of all the roots with their signs changed; the coefficient of the third term, the sum of all the products that can be formed, by taking the roots two and two under their proper signs; the coefficient of the fourth term, the sum of all the products that can be formed by taking the roots three and three with their signs changed; and so on, the roots in all odd places, preserving their signs, and in all even places, changing their signs. And as to the last term, it is always the product of all the roots.

This

This propofition, which is proved by induction, was first given by Mr Harriot, the great improver of this part of the analytic art *.

Let x reprefent any variable quantity, and let a, b, c, d, e, *&c.* reprefent any conftant quantities; if then we fuppofe, that x can be interpreted by any of thefe conftant quantities, the product $\overline{x-a} \times \overline{x-b} \times \overline{x-c} \times \overline{x-d} \times \overline{x-e}$, *&c.* will evidently vanifh, or be equal to nothing: And, hence may be derived affected equations of all degrees.

Thus: Suppofe $\overline{x-a} \times \overline{x-b} = 0$.

$$\text{Then } x^2 \left.\begin{matrix} -a \\ -b \end{matrix}\right\} x + ab = 0, \text{ a quadratic equation.}$$

Alfo, Suppofing $\overline{x-a} \times \overline{x-b} \times \overline{x-c} = 0$,

$$\text{Then } x^3 \left.\begin{matrix} -a \\ -b \\ -c \end{matrix}\right\} x^2 \left.\begin{matrix} +ab \\ +ac \\ +bc \end{matrix}\right\} x - abc = 0, \text{ a cubic equation.}$$

In like manner, fuppofing $\overline{x-a} \times \overline{x-b} \times \overline{x-c} \times \overline{x-d} = 0$.

$$\text{Then } x^4 \left.\begin{matrix} -a \\ -b \\ -c \\ -d \end{matrix}\right\} x^3 \left.\begin{matrix} +ab \\ +ac \\ +ad \\ +bc \\ +bd \\ +cd \end{matrix}\right\} x^2 \left.\begin{matrix} -abc \\ -abd \\ -bcd \end{matrix}\right\} x + abcd = 0$$

A biquadratic equation.

And fo on, for equations of higher degrees.

Here the feveral particulars mentioned in this propofition, appear by infpection: And the fame will be evident, fuppofing the figns of the quantities, a, b, c, d, *&c.* in the binomial factors, to be anyhow changed.

53. Corol. 1. Since the coefficient of the fecond term of every affected equation, is the fum of all the roots with their figns changed;

* See Harriot's Praxis Artis Analyticæ, publifhed by Mr Walter Warner, an. 1631; as alfo Dr Wallis's Algebra, Sir Ifaac Newton's Arithmetica Univerfalis, and Maclaurin's Algebra.

changed; therefore, in an affected equation, if the ſum of all the affirmative roots taken together, be equal to the ſum of all the negative roots taken together, the ſecond term will vaniſh. And conſequently, in an affected equation, wanting the ſecond term, the ſum of all the affirmative roots taken together, muſt be equal to that of all the negative roots taken together.

54. Corol. 2. In every affected equation, the coefficients of the third, fourth, fifth, *&c.* terms, are always DIVISORS of the laſt term.

55. Corol. 3. Since, in every affected equation, the laſt term is the product of all the roots; it is therefore evident, that every *commenſurate* affected equation, will have its roots among the diviſors, ſimple and compound, of the laſt term.

56. Corol. 4 Since unity is the leaſt ſimple diviſor of any quantity, therefore, if we divide any quantity by its leaſt diviſor, greater than unit, the quotient by its leaſt diviſor, greater than unit, and continue this proceſs until there be no remainder; it is evident, if we adjoin unity to theſe diviſors, we ſhall have all the ſimple diviſors of the quantity propoſed: And as for the compound diviſors, they are nothing more than the products of the ſimple diviſors taken two and two, three and three, four and four, *&c.*

57. Corol. 5. If therefore, we reſolve the laſt term of any commenſurate affected equation, into all its diviſors, ſimple and compound; we ſhall obtain the roots of this equation, ſuppoſing all the terms brought to one ſide, by obſerving what diviſors, taken affirmatively or negatively, and ſubſtituted inſtead of the variable quantity, make all the terms to vaniſh.

Ex. 1. $x^3 - 2x^2 - 5x + 6 = 0$,

Diviſors, 1, 2, 3, 6. Roots, 1, -2, 3.

Ex. 2. $x^3 - 2x^2 - 33x + 90 = 0$,

Diviſors, 1, 2, 3, 3, 5, 6, 9, 10, 15, 18, 30, 45, 90.

Roots, 3, -5, 6.

Ex. 3. $x^4 - 4x^3 - 19x^2 + 160x - 120 = 0$,

Diviſors, 1, 2, 2, 2, 3, 4, 5, 6, 8, 10, 12, 15, 20, 24, 30, 40, 60, 120. Roots, 2, 3, 4, -5.

58 Corol. 6. In any complete affected equation, involving both affirmative and negative roots, which may be known by the

 ſigns

ſigns of the terms; (ſuppoſing none of the roots to be imaginary or impoſſible quantities); the affirmative roots may be rendered negative, and the negative roots affirmative, by changing the ſigns of the alternate terms of the equation.

Ex. 1. In the equation, $x^3 - 2x^2 - 5x + 6 = 0$, the roots are, 1, -2, 3; and, in the equation, $x^3 + 2x^2 - 5x - 6 = 0$, the roots are, -1, 2, -3 (57).

Ex. 2. In the equation, $x^3 - 2x^2 - 33x + 90 = 0$, the roots are, 3, -5, 6; and, in the equation, $x^3 + 2x^2 - 33x - 90 = 0$, the roots are, -3, 5, -6 (57).

Ex. 3. In the equation, $x^4 - 4x^3 - 19x^2 + 160x - 120 = 0$, the roots are, 2, 3, 4, -5; and, in the equation, $x^4 + 4x^3 - 19x^2 - 160x - 120 = 0$, the roots are, -2, -3, -4, 5 (57.)

Ex. 4. In the equation, $x^3 + 12x^2 + 41x + 42 = 0$, the roots are, -2, -3, -7; and, in the equation, $x^3 - 12x^2 + 41x - 42 = 0$, the roots are, 2, 3, 7 (57.)

59. Corol. 7. In every complete affected equation, free from fractions, radical quantities, and imaginary roots, there are ſo many affirmative roots, as there are changes of the ſigns in a continual ſeries, from + to —, and from — to +; and for every ſucceſſion of the ſame ſigns, there are negative roots, (57, 58.)

60. Corol. 8. When there are any terms wanting in affected equations, we may ſupply the deficient places with cyphers *.

Pposition XXII.

61. The root (x) of any equation whatſoever, may be increaſed and diminiſhed, multiplied and divided, by any given quantity, (a), taken at pleaſure.

In any equation, involving a variable quantity x, if we ſuppoſe $x = y + a$, or, $y = x - a$, and ſubſtitute, it is evident, that a new equation will be produced, involving the variable quantity y, which exceeds the variable quantity x, by the given quantity a (3): And, in like manner, by ſuppoſing $y = x - a$, or, $x = y + a$, and ſubſtituting, we ſhall have a new equation, involving

* For a full illuſtration of this propoſition, and its corollaries, ſee Sir Iſaac Newton's Arithmetica Univerſalis, with Dr Wilder's Annotations; as alſo Maclaurin's Algebra, and Saunderſon's Algebra.

volving the variable quantity y, which falls ſhort of the variable quantity x, by the given quantity a (4). Alſo, if we ſuppoſe $y = ax$, or, $x = \frac{y}{a}$, and ſubſtitute, a new equation will be produced, involving the variable quantity y, a multiple of the variable quantity x, by the given quantity a: And, in like manner, by ſuppoſing $y = \frac{x}{a}$ $(= \frac{1}{a} \times x)$, or, $x = ay$, and ſubſtituting, we ſhall have a new equation, involving the variable quantity y, a multiple of the variable quantity x, by the given quantity $\frac{1}{a}$ (5.)

Let $x^3 - px^2 + qx - r = 0$ repreſent a general affected cubic equation.

Suppoſe $(x + a = y$, or,) $x = y - a$,

$$\text{(A)}\quad y^3 \left.\begin{matrix} -3a \\ -p \end{matrix}\right\} y^2 \left.\begin{matrix} +3a^2 \\ +2pa \\ +q \end{matrix}\right\} y \left.\begin{matrix} -a^3 \\ -pa^2 \\ -qa \\ -r \end{matrix}\right] = 0.$$

Suppoſe $(x - a = y$, or,) $x = y + a$,

$$\text{(B)}\quad y^3 \left.\begin{matrix} +3a \\ -p \end{matrix}\right\} y^2 \left.\begin{matrix} +3a^2 \\ -2pa \\ +q \end{matrix}\right\} y \left.\begin{matrix} +a^3 \\ -pa^2 \\ +qa \\ -r \end{matrix}\right] = 0.$$

Suppoſe $(y = ax$, or), $x = \frac{y}{a}$

$$\frac{y^3}{a^3} - \frac{py^2}{a^2} + \frac{qy}{a} - r = 0.$$

Or,

$$\text{(C)}\quad y^3 - apy^2 + a^2qy - a^3r = 0.$$

Suppoſe $(y = \frac{1}{a} \times x$, or), $x = ay$,

$$a^3y^3 - a^2py^2 + aqy - r = 0.$$

Or,

$$\text{(D)}\quad y^3 - \frac{p}{a}y^2 + \frac{q}{a^2}y - \frac{r}{a^3} = 0.$$

 62. Corol.

62. Corol. 1. By increasing and diminishing the root (x), of any affected equation, by a proper quantity (a), we may exterminate any of the intermediate terms of the equation. Thus: in the equation, (A), suppose $-3a-p=0$, or, $a=-\frac{1}{3}p$, and the second term will disappear; and, in the equation (B), supposing $3a-p=0$, or, $a=\frac{1}{3}p$, the second term vanishes also in this case. Thus: also, by supposing the coefficient of the third term equal to nothing, and resolving the quadratic equation involving a, we may exterminate the third term. In general, by increasing and diminishing the root of any affected equation, by a proper quantity (a); it will appear, that the second, third, fourth, fifth, &c. terms, may be exterminated by the resolution of a simple, an affected quadratic, affected cubic, affected biquadratic, &c. equation.

63. Corol. 2. From the last corollary, we have this general rule for exterminating the second term of any affected equation, *viz.* Divide the coefficient of the second term, by the exponent of the power of the equation, and increase or diminish the root, by this quotient, according as the second term is affirmative or negative (61.)

That is, in the equation $x^m \pm P x^{m-1}$, &c. Suppose $y \mp \frac{P}{m} = x$.

Ex. 1. $x^2 - px - q = 0$.

Suppose $y + \frac{1}{2}p = x$

$$\left.\begin{array}{rl} x^2 = & y^2 + py + \frac{1}{4}p^2 \\ -px = & \quad -py - \frac{1}{2}p^2 \\ -q = & \qquad\qquad -q \end{array}\right\}$$

$$y^2 - \frac{1}{4}p^2 - q = 0.$$

Thus: $y = \sqrt{\frac{1}{4}p^2 + q}$, and $x \left(= \frac{1}{2}p + y\right) = \frac{1}{2}p + \sqrt{\frac{1}{4}p^2 + q}$ as before, (26.)

Ex. 2. $\left.\begin{array}{l} x^3 - 3px^2 - 3q \;\; x - 2r \\ \qquad\qquad + 3p^2 x - p^3 \\ \qquad\qquad\qquad + 3pq \end{array}\right\} = 0.$

Suppose $y + p = x$, then $y^3 - 3qy - 2r = 0$.

64. Corol.

64. Corol. 3. From the equations C and D, of this proposition, it appears, that to multiply or divide the root (x), of any affected equation, by a given quantity a, we have only to range the terms, supplying deficient places with cyphers, and then multiplying or dividing the terms of the equation, by the terms of the progression, $1, a, a^2, a^3, a^4, a^5$, *&c.*

Ex. 1. Suppose we were to multiply the roots of this equation by 2, *viz.* $x^4 + 4x^3 - 19x^2 - 106x - 120 = 0$,

$$\begin{array}{lllll} x^4 + & 4x^3 - & 19x^2 - & 106x - & 120 = 0 \\ 1 & 2 & 4 & 8 & 16 \\ \hline y^4 + & 8y^3 - & 76y^2 - & 848y - & 1920 = 0, \end{array}$$

Where $y = 2x$.

Ex. 2. Let the roots of this equation, $x^3 - 3x + 1 = 0$, be multiplied by 3.

$$\begin{array}{llll} x^3 \ldots & 0x^2 \ldots & -3x + & 1 = 0 \\ 1 & 3 & 9 & 27 \\ \hline y^3 & & -27y + & 27 = 0, \end{array}$$

Where $y = 3x$.

Ex. 3. Let the roots of this equation $x^4 + 2ax^3\sqrt{2} + 8abx^2 - a^3x\sqrt{8} - 2a^2b^2 = 0$ be divided by $\sqrt{2}$.

$$\begin{array}{lllll} x^4 + & 2ax^3\sqrt{2} + & 8abx^2 - & a^3x\sqrt{8} - & 2a^2b^2 = 0 \\ 1 & \sqrt{2} & 2 & \sqrt{8} & 4 \\ \hline y^4 + & 2ay^3 + & 4aby^2 - & a^3y & -\tfrac{1}{2}a^2b^2 = 0 \end{array}$$

Where $y = \dfrac{x}{\sqrt{2}}$.

Ex. 4. Let the roots of this equation $x^3 - 84x + 160 = 0$, be divided by 2.

$$\begin{array}{llll} x^3 \ldots & 0x^2 \ldots & -84x + & 160 = 0 \\ 1 & 2 & 4 & 8 \\ \hline y^3 & & -21y + & 20 = 0, \end{array}$$

Where $y = \dfrac{x}{2}$.

65. Corol 4. By this propofition, and the laft corollary, we may change the root x of any equation, into its reciprocal, by fuppofing $y = \frac{1}{x}$, or $x = \frac{1}{y}$. Thus: in the equation.

$$x^4 - p x^3 + q x^2 - r x + s = 0,$$

$$\text{let } x = \frac{1}{y},$$

$$\text{Then } \frac{1}{y^4} - \frac{p}{y^3} + \frac{q}{y^2} - \frac{r}{y} + s = 0$$

$$\text{Or, } y^4 - \frac{r}{s} y^3 + \frac{q}{s} y^2 - \frac{p}{s} y + \frac{1}{s} = 0.$$

Where the order of the coefficients appears to be inverted by this operation.

66. Corol. 5. By increafing and diminifhing the root of an equation, by a proper quantity, we may diminifh the abfolute term, and thus render the folution of commenfurate equations more fimple and eafy (57.)

Proposition XXIII.

67. In the general equation $A + Bx + Cx^2 + Dx^3 +$, *&c.* $= a + bx + cx^2 + dx^3 +$, *&c.* where the fame powers of the fame *variable quantity* (x), are found on both fides of the equation: If we fuppofe the quantity A, to be equal to the quantity a; the coefficients of the fame powers of the variable quantity (x), will all be equal.

68. Corollary. When therefore the quantity A is *actually* equal to the quantity a, the value of the expreffion will be determined univerfally: And, when we *affume* the quantity A, equal to the quantity a, the value of the expreffion will be determined agreeable to this ASSUMPTION ONLY.

Ex. 1.

Suppofe $2Ax + 2Bx^2 + 2Cx^3 + 2Dx^4 + 2Ex^5 +$, *&c.* $= 2Ax + \overline{4B - A} \times x^2 + \overline{8C - 4B} \times x^3 + \overline{16D - 12C + B} \times x^4 + \overline{32E - 32D + 6C} \times x^5 +$, *&c.*

Here

$$\begin{aligned} \text{Here } 4B - A &= 2B \; \ldots\ldots \; B = \tfrac{1}{2}A \\ 8C - 4B &= 2C \; \ldots\ldots \; C = \tfrac{1}{3}A \\ 16D - 12C + B &= 2D \; \ldots\ldots \; D = \tfrac{1}{4}A \\ 32E - 32D + 6C &= 2E \; \ldots\ldots \; E = \tfrac{1}{5}A, \text{ \&c. \&c.} \end{aligned}$$

And, thus it appears, that either of the above expressions is equivalent to this series, *viz.*

$$2A \times \overline{x + \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 + \tfrac{1}{4}x^4 + \tfrac{1}{5}x^5 +}, \text{ \&c.}$$

Ex. 2.

Suppose $\frac{m}{n} + \frac{m}{n}Ax + \frac{m}{n}Bx^2 + \frac{m}{n}Cx^3 + \frac{m}{n}Dx^4 +$, &c. $= A + \overline{2B+A} \times x + \overline{3C+2B} \times x^2 + \overline{4D+3C} \times x^3 + \overline{5E+4D} \times x^4 +$, &c.

Here assuming $A = \frac{m}{n}$,

$$\begin{aligned} \text{Then } 2B + A &= \frac{m}{n}A \; \ldots\ldots \; B = \frac{m-n}{2n}A \\ 3C + 2B &= \frac{m}{n}B \; \ldots\ldots \; C = \frac{m-2n}{3n}B \\ 4D + 3C &= \frac{m}{n}C \; \ldots\ldots \; D = \frac{m-3n}{4n}C \\ 5E + 4D &= \frac{m}{n}D \; \ldots\ldots \; E = \frac{m-4n}{5n}D \\ &\text{\&c.} \qquad\qquad\qquad \text{\&c.} \end{aligned}$$

And thus, upon the supposition, that $A = \frac{m}{n}$, the value of either of the above expressions is equivalent to this series, *viz.*

$$\frac{m}{n} + \frac{m-n}{2n}Ax + \frac{m-2n}{3n}Bx^2 + \frac{m-3n}{4n}Cx^3 + \frac{m-4n}{5n}Dx^4 +, \text{ \&c.}$$

SCHOLIUM

SCHOLIUM.

69. This propofition is the foundation of all Sir Ifaac Newton's difcoveries in Mathematical Analyfis: And upon it depends the whole of the doctrine of infinite feries, approximations, &c.

PROPOSITION XXIV.

70. Suppofing m, n, to reprefent any affirmative integral numbers, if the quantity $v^{\frac{m}{n}} - z^{\frac{m}{n}}$, be divided by the quantity $v-z$, the quotient will be equal to

$$v^{\frac{m}{n}-1} \times \frac{1 + \frac{z}{v} + \overline{\frac{z}{v}}|^2 + \overline{\frac{z}{v}}|^3 \; (m)}{1 + \overline{\frac{z}{v}}|^{\frac{m}{n}} + \overline{\frac{z}{v}}|^{\frac{2m}{n}} + \overline{\frac{z}{v}}|^{\frac{3m}{n}} \; (n)}.$$

This is Mr Landen's Theorem, upon which he founds his RESIDUAL ANALYSIS; and it is eafily demonftrated by divifion.

71. Corol. 1. By fuppofing $v = z$, the quantity $\frac{v^{\frac{m}{n}} - z^{\frac{m}{n}}}{v - z}$ vanifhes altogether, and the feries becomes $\frac{m}{n} \times v^{\frac{m}{n}-1}$: This quantity $\frac{m}{n} \times v^{\frac{m}{n}-1}$ is therefore called the LIMIT of the fraction $\frac{v^{\frac{m}{n}} - z^{\frac{m}{n}}}{v - z}$, when both the terms are fuppofed to vanifh.

72. Corol. 2. Let the variable quantity x, be fuppofed to be increafed by the quantity w; then, when x becomes $x+w$, it is evident, that $x^{\frac{m}{n}}$ will become $\overline{x+w}|^{\frac{m}{n}}$, the increment acquired by $x^{\frac{m}{n}}$, being $\overline{x+w}|^{\frac{m}{n}} - x^{\frac{m}{n}}$, when that acquired by x, is (w, or) $\overline{x+w} - x$. If therefore, we fuppofe, $v = x + w$, and $z = x$, then

then by substitution $\frac{\overline{x+w}\rvert^{\frac{m}{n}} - x^{\frac{m}{n}}}{\overline{x+w} - x} = \overline{x+w}\rvert^{\frac{m}{n}-1} \times$

$$\frac{1 + \frac{x}{x+w} + \overline{\frac{x}{x+w}}\rvert^{2} + \overline{\frac{x}{x+w}}\rvert^{3} \ (m)}{1 + \overline{\frac{x}{x+w}}\rvert^{\frac{m}{n}} + \overline{\frac{x}{x+w}}\rvert^{\frac{2m}{n}} + \overline{\frac{x}{x+w}}\rvert^{\frac{3m}{n}} \ (n)}$$

: And, if we suppose w to vanish, the quantity $\frac{\overline{x+w}\rvert^{\frac{m}{n}} - x^{\frac{m}{n}}}{\overline{x+w} - x}$ vanishes altogether, and the series becomes $\frac{m}{n} \times x^{\frac{m}{n}-1}$, which, therefore, is the LIMIT (71) of the quantity $\frac{\overline{x+w}\rvert^{\frac{m}{n}} - x^{\frac{m}{n}}}{\overline{x+w} - x}$: Or, which is the same, the ratio $\frac{m}{n} \times x^{\frac{m}{n}-1} : 1$, is according to Sir Isaac Newton, the ULTIMATE RATIO of the increments $\overline{x+w}\rvert^{\frac{m}{n}} - x^{\frac{m}{n}}$ and $\overline{x+w} - x$.

73. Corol. 3. Suppose $v = 1+Q$ and $z = 1+R$; then by substitution, $\left(\frac{\overline{1+Q}\rvert^{\frac{m}{n}} - \overline{1+R}\rvert^{\frac{m}{n}}}{\overline{1+Q} - \overline{1+R}} =\right) \frac{\overline{1+Q}\rvert^{\frac{m}{n}} - \overline{1+R}\rvert^{\frac{m}{n}}}{Q-R}$

$$= \overline{1+Q}\rvert^{\frac{m}{n}-1} \times \frac{1 + \frac{1+R}{1+Q} + \overline{\frac{1+R}{1+Q}}\rvert^{2} + \overline{\frac{1+R}{1+Q}}\rvert^{3} \ (m)}{1 + \overline{\frac{1+R}{1+Q}}\rvert^{\frac{m}{n}} + \overline{\frac{1+R}{1+Q}}\rvert^{\frac{2m}{n}} + \overline{\frac{1+R}{1+Q}}\rvert^{\frac{3m}{n}} \ (n)}$$

Here supposing $1+Q = 1+R$, the quantity $\dfrac{\overline{1+Q}|^{\frac{m}{n}} - \overline{1+R}|^{\frac{m}{n}}}{Q-R}$ vanishes altogether, and the series becomes $\frac{m}{n} \times \overline{1+Q}|^{\frac{m}{n}-1}$.

74. Corol. 4. Suppose $\overline{1+Q}|^{\frac{m}{n}} = 1 + AQ + BQ^2 + CQ^3 + DQ^4 +$, *&c.* then $\overline{1+R}|^{\frac{m}{n}} = 1 + AR + BR^2 + CR^3 + DR^4 +$, *&c.* And therefore, $\overline{1+Q}|^{\frac{m}{n}} - \overline{1+R}|^{\frac{m}{n}} = A \times \overline{Q-R} + B \times \overline{Q^2-R^2} + C \times \overline{Q^3-R^3} + D \times \overline{Q^4-R^4} +$, *&c.* from whence we have $\dfrac{\overline{1+Q}|^{\frac{m}{n}} - \overline{1+R}|^{\frac{m}{n}}}{Q-R} = A + B \times \overline{Q+R} + C \times \overline{Q^2+QR+R^2} +$, *&c.* universally.

Here, supposing $Q=R$, the quantity $\dfrac{\overline{1+Q}|^{\frac{m}{n}} - \overline{1+R}|^{\frac{m}{n}}}{Q-R}$ vanishes altogether, and the series becomes $A + 2BQ + 3CQ^2 + 4DQ^3 +$, *&c.* $= \frac{m}{n} \times \overline{1+Q}|^{\frac{m}{n}-1}$, (73.)

75. Corol. 5. Since then $\frac{m}{n} \times \overline{1+Q}|^{\frac{m}{n}-1} = A + 2BQ + 3CQ^2 + 4DQ^3 +$, *&c.* therefore, multiplying both sides by $1+Q$ or $\overline{1+Q}|^1$ we have $\frac{m}{n} \times \overline{1+Q}|^{\frac{m}{n}} = A + \overline{2B+A} \times Q + \overline{3C+2B} \times Q^2 + \overline{4D+3C} \times Q^3 +$, *&c.* $= \frac{m}{n} + \frac{m}{n} AQ + \frac{m}{n} BQ^2 + \frac{m}{n} CQ^3 +$, *&c.* by supposition (74): whence,

taking

taking $A = \frac{m}{n}$, it is, (as in 68. *Ex.* 2.) $\overline{1+Q}\big|^{\frac{m}{n}} = 1 + \frac{m}{n}$
$Q + \frac{m}{n} \cdot \frac{m-n}{2n} Q^2 + \frac{m}{n} \cdot \frac{m-n}{2n} \cdot \frac{m-2n}{3n} Q^3 +$, &c.

76. Corol. 6. It appears from the last corollary, that $\overline{1+Q}\big|^{\frac{m}{n}}$
$= 1 + \frac{m}{n} Q + \frac{m}{n} \cdot \frac{m-n}{2n} Q^2 + \frac{m}{n} \cdot \frac{m-n}{2n} \cdot \frac{m-2n}{3n} Q^3$
$+$, &c. therefore, $\left(\frac{1}{\overline{1+Q}\big|^{\frac{m}{n}}} =\right) \overline{1+Q}\big|^{-\frac{m}{n}}$ (10) $=$

$$\frac{1}{1 + \frac{m}{n} Q + \frac{m}{n} \cdot \frac{m-n}{2n} Q^2 + \frac{m}{n} \cdot \frac{m-n}{2n} \cdot \frac{m-2n}{3n} Q^3 +, \&c.}$$

$= 1 - \frac{m}{n} Q - \frac{m}{n} \cdot \frac{-m-n}{2n} Q^2 - \frac{m}{n} \cdot \frac{-m-n}{2n} \cdot \frac{-m-2n}{3n}$
$Q^3 -$, &c. by actual division, (7, 8.) The series, in this case, as is obvious, can never terminate.

77. Corol. 7. Thus: then it appears, supposing $\frac{m}{n}$ to be any number, integral or fractional, affirmative or negative; that
$\overline{1+Q}\big|^{\frac{m}{n}} = 1 + \frac{m}{n} Q + \frac{m}{n} \cdot \frac{m-n}{2n} Q^2 + \frac{m}{n} \cdot \frac{m-n}{2n} \cdot$
$\frac{m-2n}{3n} Q^3 +$, &c. which is Sir Isaac Newton's famous binomial theorem, investigated universally, according to Mr Landen's method * (75, 76).

78. Corol. 8. Let $P + PQ$ represent any binomial, then, since $\overline{P+PQ}\big|^{\frac{m}{n}} = P^{\frac{m}{n}} \times \overline{1+Q}\big|^{\frac{m}{n}}$ (5), therefore, by the last corollary

* Discourse concerning the Residual Analysis, pages 6th and 7th.

rollary (77), $\overline{P+PQ}|^{\frac{m}{n}} = P^{\frac{m}{n}} + \frac{m}{n}\overset{A}{A}Q + \frac{m-n}{2n}\overset{B}{B}Q + \frac{m-2n}{3n}\overset{C}{C}Q + \frac{m-3n}{4n}\overset{D}{D}Q +$, &c. Here P is the first term, Q second divided by the first, $\frac{m}{n}$, the exponent, and A, B, C, D, E, &c. the foregoing terms with their proper signs; and this is the general formula, called the *Binomial Theorem*, as given by the illustrious inventor, Sir Isaac Newton, in his letter to Mr Oldenburgh, 13th June 1676; which letter, was transmitted by Mr Oldenburgh to M. Leibnitz, on the 26th of June 1676 *. By this most excellent theorem, we can multiply and divide, involve and evolve all quantities, whether terminate or interminate. We shall illustrate this theorem by some easy examples.

Ex. 1. Suppose we were to raise 18 to the 4th power. Here $18 = 10 + 8$; and so $P = 10$, $Q = \frac{8}{10} = \frac{4}{5}$, and $\frac{m}{n} = \frac{4}{1}$, or $m = 4$, and $n = 1$.

$$P^{\frac{m}{n}} \ldots\ldots\ldots\ldots\ldots\ldots = 10000 = A$$

$$\frac{m}{n}AQ = 4 \times 10000 \times \frac{4}{5} \ldots\ldots\ldots = 32000 = B$$

$$\frac{m-n}{2n}BQ = \frac{3}{2} \times 32000 \times \frac{4}{5} \ldots\ldots = 38400 = C$$

$$\frac{m-2n}{3n}CQ = \frac{2}{3} \times 38400 \times \frac{4}{5} \ldots\ldots = 20480 = D$$

$$\frac{m-3n}{4n}DQ = \frac{1}{4} \times 20480 \times \frac{4}{5} \ldots\ldots = 4096 = E$$

$$\overline{P+PQ}|^{\frac{m}{n}} = \overline{18}|^{4} \ldots\ldots\ldots\ldots = 104976$$

* See Commercium Epistolicum, p. 49.—57; as also Dr Wallis's Algebra.

And

And the ſame may be derived by dividing 18 into any other two parts. Thus: Since $18 = 6 + 12$; therefore, $P = 6$, $Q = \frac{12}{6} = 2$

$$\overline{P}^{\frac{m}{n}} = \overline{6}|^{4} \ldots\ldots\ldots\ldots = 1296 = A$$

$$\frac{m}{n} A Q \ldots\ldots\ldots\ldots = 10368 = B$$

$$\frac{m-n}{2n} B Q \ldots\ldots\ldots\ldots = 31104 = C$$

$$\frac{m-2n}{3n} C Q \ldots\ldots\ldots\ldots = 41472 = D$$

$$\frac{m-3n}{4n} D Q \ldots\ldots\ldots\ldots = 20736 = E$$

$$\overline{P+PQ}|^{\frac{m}{n}} = \overline{18}|^{4} \ldots\ldots\ldots\ldots = 104976$$

Ex. 2. To extract the ſquare root of the number 2. Here $2 = 1 + 1$; whence, $P = 1$, $Q = 1$, and $\frac{m}{n} = \frac{1}{2}$, or $m = 1$, and $n = 2$.

$$P^{\frac{m}{n}} \ldots\ldots\ldots\ldots = 1 = A$$

$$\frac{m}{n} A Q = \frac{1}{2} \times 1 \ldots\ldots\ldots\ldots = \frac{1}{2} = B$$

$$\frac{m-n}{2n} B Q = -\frac{1}{4} \times \frac{1}{2} \ldots\ldots\ldots = -\frac{1}{2.4} = C$$

$$\frac{m-2n}{3n} C Q = -\frac{3}{6} \times -\frac{1}{2.4} \ldots\ldots = \frac{1.3}{2.4.6} = D$$

$$\frac{m-3n}{4n} D Q = -\frac{5}{8} \times \frac{1.3}{2.4.6} \ldots\ldots = -\frac{1.3.5}{2.4.6.8} = E$$

&c. &c.

Here the law of the ſeries is obvious; and $\sqrt{2} = 1\frac{1}{2} - \frac{1}{2.4} + \frac{1.3}{2.4.6} - \frac{1.3.5}{2.4.6.8} + \frac{1.3.5.7}{2.4.6.8.10} -$ &c.

Ex. 3.

Ex. 3. To extract the square root of $a^2 - x^2$. Here $P = a^2$, $Q = -\frac{x^2}{a^2}$, $\frac{m}{n} = \frac{1}{2}$, or $m = 1$, and $n = 2$.

$$P^{\frac{m}{n}} = \overline{a^2}|^{\frac{1}{2}} \quad \ldots\ldots\ldots\ldots \quad = a = A$$

$$\frac{m}{n} A Q = \frac{1}{2} a \times -\frac{x^2}{a^2} \quad \ldots\ldots \quad = -\frac{x^2}{2.a} = B$$

$$\frac{m-n}{2n} B Q = -\frac{1}{4} \times -\frac{x^2}{2.a} \times -\frac{x^2}{a^2} \;. = -\frac{x^4}{2.4.a^3} = C$$

$$\frac{m-2n}{3n} C Q = -\frac{3}{6} \times -\frac{x^4}{2.4.a^3} \times -\frac{x^2}{a^2} = -\frac{3.x^6}{2.4.6a^5} = D$$

&c. &c.

Here the law of the series appears; and $\sqrt{a^2 - x^2}$, or $\overline{a^2 - x^2}|^{\frac{1}{2}}$ $= a - \frac{x^2}{2a} - \frac{x^4}{8a^3} - \frac{x^6}{16a^5} - \frac{5x^8}{128a^7} - \frac{7x^{10}}{256a^9} -$, &c.

Ex. 4. To determine the value of $\frac{ay}{\sqrt{a^2 - y^2}}$ in a series. Here $\frac{ay}{\sqrt{a^2 - y^2}} = ay \times \overline{a^2 - y^2}|^{-\frac{1}{2}}$; whence, $P = a^2$, $Q = -\frac{y^2}{a^2}$, $\frac{m}{n} = -\frac{1}{2}$, or $m = -1$, and $n = 2$.

$$P^{\frac{m}{n}} = a^{2 \times -\frac{1}{2}} = a^{-1} \quad \ldots\ldots\ldots \quad = \frac{1}{a} = A$$

$$\frac{m}{n} A Q = -\frac{1}{2} \times \frac{1}{a} \times -\frac{y^2}{a^2} \quad \ldots\ldots \quad = \frac{y^2}{2, a^3} = B$$

$$\frac{m-n}{2n} B Q = -\frac{3}{4} \times \frac{y^2}{2.a^3} \times -\frac{y^2}{a^2} \quad \ldots \quad = \frac{3 \cdot y^4}{2.4.a^5} = C$$

$$\frac{m-2n}{3n} C Q = -\frac{5}{6} \times \frac{3 \cdot y^4}{2.4.a} \times -\frac{y^2}{a^2} \;.. = \frac{3.5.y^6}{2.4.6.a^7} = D$$

&c. &c.

Here

Here the law of the series is obvious; and $\overline{a^2 - y^2}\big|^{-\frac{1}{2}} = \frac{1}{a} + \frac{y^2}{2a^3} + \frac{3y^4}{8a^5} + \frac{5y^6}{16a^7} +$, &c. and therefore,

$$\left(ay \times \overline{a^2 - y^2}\big|^{-\frac{1}{2}} =\right) \frac{ay}{\sqrt{a^2 - y^2}} = y + \frac{y^3}{2a^2} + \frac{3y^5}{8a^4} + \frac{5y^7}{16a^6} +, \text{ \&c.}$$

Ex. 5. $\sqrt[3]{1 - x^3} = \overline{1 - x^3}\big|^{\frac{1}{3}} = 1 - \frac{x^3}{3} - \frac{x^6}{9} - \frac{5x^9}{81} - \frac{10x^{12}}{243} - \frac{22x^{15}}{729} -$, &c.

Ex. 6. $\sqrt[3]{\frac{a^2}{\overline{a^2 + x^2}\big|^2}} = a^{\frac{2}{3}} \times \overline{a^2 + x^2}\big|^{-\frac{2}{3}} =$

$$\frac{1}{\sqrt[3]{a^2}} \times \overline{1 - \frac{2x^2}{3a^2} + \frac{5x^4}{9a^4} - \frac{40x^6}{81a^6} +, \text{ \&c.}}$$

Ex. 7. $\sqrt[5]{a^2 - x^2} = \overline{a^2 - x^2}\big|^{\frac{1}{5}} =$

$$a^{\frac{2}{5}} \times \overline{1 - \frac{x^2}{5a^2} - \frac{2x^4}{25a^4} - \frac{6x^6}{125a^6} - \frac{21x^8}{625a^8} -, \text{ \&c.}}$$

79. Corol. 9. The theorem given in the last corollary, may be applied to any quantity, terminate or interminate, either by substitution, or by connecting the terms properly with the vinculum. And, hence may be derived Sir Isaac Newton's two theorems for the reversion of series, and all others of the same kind. Thus:

Case I. Suppose $v = Az + Bz^2 + Cz^3 + Dz^4 + Ez^5 +$, &c.

Then $z = \frac{1}{A}v - \frac{B}{A^3}v^2 + \frac{2B^2 - AC}{A^5}v^3 + \frac{5ABC - A^2D - 5B^3}{A^7}v^4 + \frac{14B^4 - 21AB^2C + 6A^2BD + 3A^2C^2 - A^3E}{A^9}v^5 +$, &c.

Case

Case II. Suppose $v = Az + Bz^3 + Cz^5 + Dz^7 +$, &c.

Then $z = \frac{1}{A}v - \frac{B}{A^4}v^3 + \frac{3B^2 - AC}{A^7}v^5 + \frac{8ABC - A^2D - 12B^3}{A^{10}}v^7$

$$+ \frac{55B^4 - 55AB^2C + 10A^2BD + 5A^2C^2 - A^3E}{A^{13}}v^9$$

$+$, &c.

Case III. Suppose $v = Az^{m} + Bz^{m+n} + Cz^{m+2n} + Dz^{m+3n} +$, &c. Then (supposing $u = \frac{v}{A}$) $z = u^{\frac{1}{m}}$

$$- \frac{B}{mA}u^{\frac{1+n}{m}} + \frac{\overline{m+1+2n}\,B^2 - 2mAC}{2m^2A^2}u^{\frac{1+2n}{m}}$$

$$\left.\begin{array}{l} -\overline{2m^2+9mn+9n^2+3m+6n}\,B^3 \\ \qquad + \frac{\overline{m+3n+1}}{m^2A^2}BC \\ \qquad - \frac{D}{mA} \end{array}\right\} u^{\frac{1+3n}{m}} +, \text{\&c.}$$

Case IV. Suppose $av + bv^2 + cv^3 + dv^4 +$, &c. $=$ $gz + hz^2 + kz^3 + lz^4 +$, &c. Then $v = \frac{g}{a}v +$

$$\frac{h - bA^2}{a}v^2 + \frac{k - 2bAB - cA^3}{a}v^3 +$$

$$\frac{l - bB^2 - 2bAC - 3cA^2B - dA^4}{a}v^4 +, \text{\&c.}$$

Where A, B, C, D, &c. are the coefficients of the first, second, third, &c. terms.

Ex. 1. Suppose $v = z - \frac{1}{2}z^2 + \frac{1}{3}z^3 - \frac{1}{4}z^4 +$, &c.

Then (Case I.) $z = v + \frac{v^2}{1.2} + \frac{v^3}{1.2.3} + \frac{v^4}{1.2.3.4} + \frac{v^5}{1.2.3.4.5}$

$+$, &c.

Ex. 2.

Ex. 2. Suppose $v = z + \frac{1}{6}z^3 + \frac{3}{40}z^5 + \frac{5}{112}z^7 +$, &c.

Then (Case II.) $z = v - \frac{v^3}{1.2.3} + \frac{v^5}{1.2.3.4.5} - \frac{v^7}{1.2.3.4.5.6.7} +$, &c.

Ex. 3. Suppose $v = z - \frac{a^2}{2z} + \frac{a^4}{6z^3} - \frac{a^6}{24z^5} +$, &c.

Then (Case III.) $z = v + \frac{a^2}{2v} - \frac{5a^4}{12v^3} + \frac{5a^6}{8v^5}$, &c.

Ex. 4. Suppose $v - \frac{1}{2}v^2 + \frac{1}{6}v^3 - \frac{1}{24}v^4$, &c.
$= \frac{1}{2}z + \frac{1}{3}z^2 + \frac{1}{4}z^3 + \frac{1}{5}z^4$, &c.

Then (Case IV.) $v = \frac{1}{2}z + \frac{11}{24}z^2 + \frac{11}{24}z^3 + \frac{1381}{2880}z^4$, &c.

Proposition XXV *.

80. In the single equation $y^m = N$, suppose $\frac{A}{B}$ to be a near approximation to the value of y; then, by continually substituting the approximating quantities into this general formula

$$\frac{N + \overline{m-1} \times \overline{\frac{A}{B}}\big|^{m}}{m \times \overline{\frac{A}{B}}\big|^{m-1}}$$

The value of y may be determined to *any degree of accuracy*. Or, universally let $ay^m + by^{m-1} + cy^{m-2} + dy^{m-3}$, &c. $= 0$, represent any affected equation; and, suppose $\frac{A}{B}$ to be a near approximation to the value of y; then, by continually substituting the approximating quantities into the following general formula, the value of y may be determined to *any degree of accuracy*.

General Formula.

$$\frac{\overline{m-1}\,aA^m + \overline{m-2}\,bA^{m-1}B + \overline{m-3}\,cA^{m-2}B^2 +, \&c.}{maA^{m-1}B + \overline{m-1}\,bA^{m-2}B^2 + \overline{m-2}\,cA^{m-3}B^3 +, \&c.}$$

I This

* See Maclaurin's Algebra, Simpson's Algebra and Select Exercises, as also Emerson's Algebra.

This propofition is demonftrated directly by Sir Ifaac Newton's Binomial Theorem, (78) *, and the general formula will refolve any affected equation whatfoever. Thus: in the quadratic equation, $y^2 + by = c$, the approximating quantity is $\frac{A^2 + cB^2}{\overline{2A + bB} \times B}$; in the cubic equation, $y^3 + by^2 + cy = d$, the approximating quantity is $\frac{\overline{2A + bB} \times A^2 + dB^3}{\overline{3A + 2bAB + cB^2} \times B}$; in the biquadratic equation, $y^4 + by^3 + cy^2 + dy = c$, the approximating quantity is $\frac{\overline{3A^2 + 2bAB + cB^2} \times A^2 + cB^4}{\overline{4A^3 + 3bA^2B + 2cAB^2 + dB^3} \times B}$; and thus, from the general formula, you may derive the approximating theorem for any equation.

Ex. 1. Suppofe $y^2 = 2$.

Here taking $\frac{A}{B} = 1 \left(= \frac{1}{1}\right)$ for the firft approach to the root; we have $y = \frac{3}{2}$ for a nearer approximation. *Again*, taking $\frac{A}{B} = \frac{3}{2}$ we have $y = \frac{17}{12}$ for a third approximation. Alfo, taking $\frac{A}{B} = \frac{17}{12}$, we have $y = \frac{3462}{2448}$ for a fourth approximation. And taking $\frac{A}{B} = \frac{3462}{2448}$, we fhall obtain a fifth approximation, which being reduced, will give $y = 1.414213562378$, nearly.

Ex. 2. Suppofe $y^2 = 12$.

Firft approximation, $\frac{A}{B} (= 3) = \frac{3}{1}$.

Second,——— $\frac{A}{B} \left(= \frac{9 + 12}{6}\right) = \frac{7}{2}$.

* See Mr Colfon's Commentary on Sir Ifaac Newton's Method of Fluxions, p. 186. — 191.

Third

Third, —— $\frac{A}{B}\left(=\frac{49+48}{14\times 2}\right)=\frac{97}{28}$.

Fourth, —— $\frac{A}{B}\left(=\frac{\overline{97}|^2+12\times\overline{28}|^2}{194\times 28}\right)=\frac{18817}{5432}$.

Fifth, —— $\frac{A}{B}\left(=\frac{\overline{18817}|^2+12\times\overline{5432}|^2}{37634\times 5432}\right)=\frac{708158977}{204427888}=$ $3.4641016151377545 9$, nearly.

Ex. 3. Suppose $y^3-2y=5$.

Here $a=1$, $b=0$, $c=-2$, and $d=5$.

First approach, $\frac{A}{B}\ (=2)=\frac{2}{1}$.

Second, —— $\frac{A}{B}\left(=\frac{16+5}{10}\right)=\frac{21}{10}$.

Third, —— $\frac{A}{B}\left(=\frac{9261-2500}{5615}\right)=\frac{11761}{5615}$.

Fourth, —— $\frac{A}{B}\left(=\frac{2\times\overline{11761}|^3+5\times\overline{5615}|^3}{3\times\overline{11761}|^2\times 5615-2\times\overline{5615}|^3}\right)=$ $\frac{4138744325037}{1975957316495}=2.09455148170 1$, nearly.

Ex. 4. Suppose $y^3+pxy+p^2y-x^3-2p^3=0$, to determine the value of y in a converging series, supposing x to be small in comparison of y.

Since x is supposed to be small in comparison of y, that the series may converge; therefore, to determine the first approximation, suppose x to vanish, and we have $y^3+p^2y-2p^3=0$, which gives $y=p\ (57)=\frac{A}{B}$. Then, reducing the equation to a proper form, *viz.*

$$y^3+\overline{p^2+px}\times y-\overline{2p^3+x^3}\times y^0=0,$$

It appears directly, that $m=3$, $a=1$, $b=0$, $c=p^2+px$, $d=-2p^3-x^3$, $e=0$. Thus:

First approach, $\frac{A}{B}\ (=p)=\frac{p}{1}$.

2d, —— $\frac{A}{B} = \frac{4p^3 + x^3}{4p^2 + px}$.

3d, —— $\frac{A}{B} = \frac{256p^9 + 96p^8x + 24p^7x^2 + 114p^6x^3 + 48p^5x^4 +, \&c.}{256p^8 + 160p^7x + 60p^6x^2 + 109p^5x^3 + 25p^4x^4 +, \&c.}$

$= p - \frac{1}{4}x + \frac{x^2}{64p} + \frac{131x^3}{512p^2} + \frac{509x^4}{16384p^3}$, &c.

81. Corol. 1. Although this proposition is general, yet in resolving literal equations (*Ex.* 4.) the following method will often be found more easy. In a literal affected equation, involving two variable quantities, x, y, assume $y = Ax^n + Bx^{n+r} + Cx^{n+s} + Dx^{n+t}$, &c. And, instead of y and its powers, in the proposed equation, substitute Ax^n and its powers; then, to determine n, suppose the two least exponents equal for an ascending series, or the two greatest for a descending one: Substitute the value of n thus obtained, into all these exponents, and having taken the least for an ascending series, or the greatest for a descending one; subtract it from each of the rest, and the remainders added to themselves, and to one another, all *possible* ways, will give the values of r, s, t, &c. And as for the coefficients, A, B, C, D, &c they will be obtained by Prop. XX. (67, 68.)

Ex. 1. $p^4x^2 - p^4xy + x^6 - py^5 = 0$,

Here $p^4x^2 - p^4Ax^{n+1} + x^6 - pA^5x^{5n} = 0$.

Supposing $n + 1 = 2$, we have $n = 1$,

Exponents, 2, 2, 6, 5,

Remainders, 3, 4, which give the numbers, 3, 4, 6, 7, 8, 9, 10, &c. for r, s, t, &c.

Thus: For an ascending series, $y = Ax + Bx^4 + Cx^5 + Dx^7 +$, &c.

$$\left.\begin{array}{l} p^4x^2 = p^4x^2 \\ -p^4xy = -p^4Ax^2 - p^4Bx^5 - p^4Cx^6 - p^4Dx^8 \\ +\ x^6 = \ .\ .\ .\ .\ .\ .\ +\ x^6 \\ -py^5 = \ .\ .\ -pA^5x^5\ .\ -5pA^4Bx^8, \&c. \end{array}\right\} = 0.$$

Therefore,

Therefore, equating the homologous terms, (68).

$$\begin{array}{ll} p^4 - p^4 A = 0 \quad \ldots\ldots\ldots\ldots & A = 1 \\ -p^4 B - p A^5 = 0 \quad \ldots\ldots\ldots\ldots & B = -\frac{1}{p^3} \\ -p^4 C + 1 = 0 \quad \ldots\ldots\ldots\ldots & C = \frac{1}{p^4} \\ -p^4 D - 5 p A^4 B = 0 \quad \ldots\ldots\ldots\ldots & D = \frac{5}{p^6} \\ \quad \&c. & \&c. \end{array}$$

Hence, $y = x - \frac{x^4}{p^3} + \frac{x^5}{p^4} + \frac{5x^7}{p^6}$, &c.

If we suppose $5n = 6$, or $n = 1\frac{1}{5}$, we shall have for a descending series, $y = A x^{1\frac{1}{5}} + B x^{-2\frac{3}{5}} + C x^{-2\frac{4}{5}} + D x^{-6\frac{2}{5}}$, &c. $= \frac{x^{1\frac{1}{5}}}{p^{\frac{1}{5}}} - \frac{p^{3\frac{3}{5}}}{5x^{2\frac{3}{5}}} + \frac{p^{3\frac{4}{5}}}{5x^{2\frac{4}{5}}}$, &c. viz. by proceeding as above.

Ex. 2. $y^3 + p^2 y + p x y - x^3 - 2p^3 = 0$,

$$A^3 x^{3n} + p^2 A x^n + p A x^{n+1} - x^3 - 2p^3 x^0 = 0.$$

Suppose $n = 0$, for an ascending series.

Series for r, s, t, &c. 1, 2, 3, 4, 5, &c.

And $y = A + Bx + Cx^2 + Dx^3 + Ex^4$, &c.

$$\left.\begin{array}{rl} y^3 = & A^3 + 3A^2Bx + 3AB^2x^2 + B^3x^3, \&c. \\ & \qquad\qquad + 3A^2Cx^2 + 6ABCx^3 \\ & \qquad\qquad\qquad\qquad + 3A^2Dx^3 \\ +p^2y = & +p^2A + p^2Bx + p^2Cx^2 + p^2Dx^3, \&c. \\ +pxy = & \;.\;.\; + pAx + pBx^2 + pCx^3, \&c. \\ -x^3 = & \ldots\ldots\ldots\ldots - x^3 \\ -2p^3 = & -2p^3 \end{array}\right\} = 0$$

Hence,

Hence, (68), $A^3 + p^2 A - 2p^3 = 0 \quad \ldots \quad A = p$ (57),

$$3A^2B + p^2B + pA = 0 \quad \ldots \quad B = -\frac{1}{4}$$

$$3AB^2 + 3A^2C + p^2C + pB = 0 \quad \ldots \quad C = \frac{1}{64p}$$

$$B^3 + 6ABC + 3A^2D + p^2D + pC - 1 = 0 \quad . \quad D = \frac{131}{512p^2}$$

&c. &c.

Thus: $y = p - \frac{1}{4}x + \frac{x^2}{64p} + \frac{131x^3}{512p^2} + \frac{509x^4}{16384p^3}$, &c. as before. And this series converges so much the faster, the less the quantity x is in comparison of p.

If we suppose $3n = 3$, or $n = 1$, we shall have $-1, -2, -3, -4, -5$, &c. for r, s, t, &c. And hence, the descending series, $y = Ax + B + Cx^{-1} + Dx^{-2}$, &c. $= x - \frac{1}{3}p - \frac{p^2}{3x} + \frac{55p^3}{81x^2} + \frac{64p^4}{243x^3}$, &c. viz. by proceeding as above. Which series, as is obvious, will converge so much the faster, the greater x is in comparison of p.

Ex. 3. $y^3 + y^2 + y - x^3 = 0$,

$$A^3x^{3n} + A^2x^{2n} + Ax^n - x^3 = 0.$$

Supposing $3n = 3$, or $n = 1$.

Then $-1, -2, -3$, &c. $= r, s, t$, &c. And,

Hence, $y = Ax + B + Cx^{-1} + Dx^{-2}$, &c.

$$\left.\begin{array}{l}
y^3 = A^3x^3 + 3A^2Bx^2 + 3A^2Cx + \left.\begin{array}{r} 3A^2D \\ + 3AB^2x + 6ABC \\ + B^3 \end{array}\right\} \&c. \\
+ y^2 = \qquad + A^2x^2 + 2ABx + \left.\begin{array}{r} B^2 \\ + 2AC \end{array}\right\} \&c. \\
+ y = \ldots\ldots\ldots\ldots + Ax + B \\
- x^3 = -x^3
\end{array}\right\} = 0.$$

Then,

Then (68), $A^3 - 1 = 0$ . . . . $A = 1$

$3A^2B + A^2 = 0$ . . . . $B = -\frac{1}{3}$

$3A^2C + 3AB^2 + 2AB + A = 0$ . . . $C = -\frac{2}{9}$

$3A^2D + 6ABC + B^3 + B^2 + 2AC + B = 0$. $D = \frac{7}{81}$

&c. &c.

Thus: $y = x - \frac{1}{3} - \frac{2}{9x} + \frac{7}{81x^2}$ &c. And this series, as is obvious, will converge so much the faster, the greater we suppose x to be takeu

If we suppose $n = 3$, then $y = Ax^3 + Bx^6 + Cx^9$, &c. for an ascending series; also, if we suppose $2n = n$, or $n = 0$, then $y = A + Bx^3 + Cx^6$, &c. which will give an ascending series also.

82. Corol. 2. Having substituted the value of n into all the exponents, as in the last corollary, let each of the exponents be subtracted from the next greater for a series of *differences*, then, if the least difference be divided by the *greatest common measure* of all the differences, the *least* number which can be divided by the quotient (40), will be such a divisor of the least difference, as will give the *common* difference of all the exponents in the series for y. Supposing, therefore, this common difference to be represented by r; we shall have the following series for y, *viz.*

$$y = Ax^n + Bx^{n+r} + Cx^{n+2r} + Dx^{n+3r}, \text{ \&c.}$$

Where the common difference r must be taken as affirmative or negative, according as the series is to ascend or descend (81.)

Ex. 1. $y^3 - p^2y + pxy - x^3 - 2p^3x^0 = 0$,

$3n$, n, $n+1$, 3, 0. Suppose $n = 0$.

Exponents, 0, 0, 1, 3, 0.

Differences, 1, 2, whereof the greatest common measure is, $1 \frac{1}{1} = 1$, and $r = 1$, (81.)

Hence, $y = A + Bx + Cx^2 + Dx^3$, &c. as before.

If we suppose $3n = 3$, or $n = 1$.

Exponents 3, 1, 2, 3, 0.

Differences, 1, 1. Greatest common measure, 1.

Hence, $r = -1$, (81.) and

$y = Ax + B + Cx^{-1} + Dx^{-2}$, &c. as before.

Ex. 2.

Ex. 2. $y^5 - py^2 + 9px^2 - x^3 = 0$

$5n,\ 2n,\ 2,\ 3.$ Suppose $2n = 2$; or $n = 1$,

Exponents, 5, 2, 2, 3.

Differences, 1, 2. Greatest common measure 1.

Hence, $r = 1$, and

$$y = Ax + Bx^2 + Cx^3 + Dx^4, \ \&c.$$

If we suppose $5n = 3$, or $n = \frac{3}{5}$,

Exponents, 3, $1\frac{1}{5}$, 2, 3.

Differences, $\frac{4}{5}$, 1. Greatest common measure, 1.

The least number, divisible by $\frac{4}{5}$, is 4, (40.) And $\frac{4}{5}$ divided by 4, gives $\frac{1}{5}$ for the quotient.

Hence, $r = -\frac{1}{5}$, and

$$y = Ax^{\frac{3}{5}} + Bx^{\frac{2}{5}} + Cx^{\frac{1}{5}} + D, \ \&c.$$

Ex. 3. $y^3 - pxy + x^3 = 0$,

$3n,\ n+1,\ 3.$ Suppose $n + 1 = 3$, or $n = 2$.

Exponents, 6, 3, 3.

Differences, 3. Greatest common measure 3.

$\frac{3}{3} = 1$, and $\frac{3}{1} = 3 = r$.

Hence, $y = Ax^2 + Bx^5 + Cx^8$, &c.

If we suppose $3n = 3$, or $n = 1$.

Exponents, 3, 2, 3.

Differences, 1. Greatest common measure, 1.

$\frac{1}{1} = 1$, and $r = -1$.

Hence, $y = Ax + B + Cx^{-1}$, &c.

83. Corol. 3. If it appears that A hath various values, some of which are equal to one another, it is obvious, that the *number* of equal values will vary according to the particular value we take of A. Thus: If a, a, a, b, b, c, represent six values of A, the number of equal values of A will be 3, 2, or 1, according as we take A equal to a, b, or c. Therefore having, as before determined, the value of n, let the differences of the exponent by which n was determined, and the other exponents, be *now* taken as

as a series of differences; let the least difference be divided by the greatest common measure of all the differences, and the least number, (45), that can be divided by this quotient, and the number of equal values of A, will be, in *every case*, the divisor of the least difference, and the quotient the *true* value of r.

Ex. 1. $x^5 - 4y^{\frac{1}{2}}x^4 + 6yx^3 - 4y^{\frac{3}{2}}x^2 + y^2x + 10x^{\frac{1}{3}} - 6y^{\frac{1}{2}} = 0$,

$5,\ \frac{1}{2}n + 4,\ n + 3,\ \frac{3}{2}n + 2,\ 2n + 1,\ \frac{1}{3},\ \frac{1}{2}n.$

Supposing $2n + 1 = 5$, or $n = 2$.

Exponents, $5,\ 5,\ 5,\ 5,\ 5,\ \frac{1}{3},\ 1$.

Differences, $4\frac{2}{3},\ 4$. Greatest common measure, $\frac{2}{3}$.

$4 \div \frac{2}{3} = 4 \times \frac{3}{2} = 6.$

In this example, A has 4 equal values. And the least number, divisible by 6, 4, is 12, (45.)

$\frac{4}{12} = \frac{1}{3}$, and therefore, $r = -\frac{1}{3}$.

Hence, $y = Ax^2 + Bx^{\frac{5}{3}} + Cx^{\frac{4}{3}} + Dx + Ex^{\frac{1}{3}}$, &c.

Ex. 2. $-\dfrac{y^9}{a^5} + xy^3 - 2x^2y^2 + x^3y + \dfrac{x^{14}}{b^{10}} = 0$,

$9n,\ 3n + 1,\ 2n + 2,\ n + 3,\ 14.$

Supposing $2n + 2 = n + 3$, or $n = 1$,

Exponents, $9,\ 4,\ 4,\ 4,\ 14$.

Differences, $5,\ 10$. Greatest common measure, 5.

$5 \div 5 = 1.$

In this example, A has 2 equal values. And the least number, divisible by 1, 2, is 2, (45.)

$\frac{5}{2} = 2\frac{1}{2}$, or $r = 2\frac{1}{2}$.

Hence, $y = Ax + Bx^{3\frac{1}{2}} + Cx^6 + Dx^{8\frac{1}{2}}$, &c.

Ex. 3. $a^4y^2 - 2a^4xy + a^4x^2 + x^4y^2 = 0$,

$2n,\ n + 1,\ 2,\ 2n + 4$. Suppose $2n = 2$, or $n = 1$.

Exponents, $2,\ 2,\ 2,\ 6$.

Differences, 4. Greatest common measure, 4.

$\frac{4}{4} = 1$. And A has 2 equal values.

The least number, divisible by 1, 2, is 2, (45.)

$\frac{4}{2} = 2 = r$.

Hence, $y = Ax + Bx^3 + Cx^5 + Dx^7$, &c. *

84. Corol. 4. General forms, for approximating infinite converging series, may easily be derived from this proposition, and *Sir Isaac Newton's* BINOMIAL THEOREM, (78.) Thus: let

$$ax^m + bx^{m+n} + cx^{m+2n} + dx^{m+3n}, \text{ \&c.}$$

represent any infinite converging series, and let the following general expressions be assumed as approximations to the value of the proposed series, *viz.*

$\dfrac{ax^m}{1 + Px^n}$, First approximation.

$\dfrac{ax^m + Ax^{m+n}}{1 + Px^n}$, Second approximation.

$\dfrac{ax^m + Ax^{m+n}}{1 + Px^n + Qx^{2n}}$, Third approximation.

Then by substitution, and comparing the homologous terms, (68), we have

$x^m \times \dfrac{\dfrac{a^2}{1 - bx^n}}{a}$, First approximation, where $P = -\dfrac{b}{a}$.

$x^m \times \dfrac{ab + \overline{b^2 - ac} \times x^n}{b - cx^n}$, Second approximation, where $P = -\dfrac{c}{b}$, and $A = b - \dfrac{ac}{b}$.

$x^m \times \dfrac{a + \overline{b + aP} \times x^n}{1 + Px^n + Qx^{2n}}$, Third approximation, where $P = \dfrac{bc - ad}{ac - b^2}$, $Q = \dfrac{bd - c^2}{ac - b^2}$, and $A = \dfrac{b \times \overline{ac - b^2} + a \times \overline{bc - ad}}{ac - b^2}$ †.

* See Sir Isaac Newton's Analysis of Equations, consisting of an infinite number of terms, with Mr Colson's Commentary on Sir Isaac Newton's Fluxions, s'Gravesande's Algebra, Stirling on Series, Maclaurin's Algebra, and Emerson's Algebra.

† See Mr Simpson's Dissertations, p. 99.

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We may also approximate any infinite converging series, by assuming a fraction, whereof one of the terms is the first term of the proposed series, and the other consists of quantities to be determined by supposing the fraction equal to two or more of the initial terms of the series.

Ex. 1. Suppose $2ax - \frac{5ab}{2}x^2 + \overline{3ab^2 - \frac{8a^3}{3}} \times x^3$, &c. to be an infinite converging series, the quantity x being small,

Assume $\left(\frac{2ax}{A-B} =\right) 2ax - \frac{5ab}{2} = \frac{2ax}{A} + \frac{2axB}{A^2}$ . . . . ; then comparing the homologous terms, (68), we have $A = 1$, and $B = -\frac{5bx}{2}$. Thus $\frac{2ax}{1+\frac{5bx}{2}}$ is the first approximating fraction. By division, $\frac{2ax}{1+\frac{5bx}{2}} = 2ax - \frac{5ab}{2}x^2 + \frac{25ab^2}{8}x^3 - \frac{125ab^3}{32}x^4$, &c. therefore, supposing the proposed series to be represented by M, by subtraction, we have $\frac{2ax}{1+\frac{5bx}{4}} - \frac{8a^3}{3}x^3 + \frac{125ab^3}{32}x^4$, &c. $=$ M, or,

$$\frac{2ax}{1+\frac{5bx}{4}} - \frac{1}{3}\times\overline{2ax}^3 - 6ax\times\overline{\frac{5bx}{4}}^3 \text{ \&c.} = M.$$

Hence, to determine a second approximating fraction,

Assume $\left(\overline{\frac{2ax}{C+D}}^3 =\right) \overline{2ax}^3 - 6ax\times\overline{\frac{5bx}{4}}^3 = \frac{\overline{2ax}^3}{C^3} - \frac{3D\times\overline{2ax}^3}{C^4}$ . . . . ; then, comparing the homologous terms, we have $C = 1$, and $3D\times\overline{2ax}^3 = 6ax\times\overline{\frac{5bx}{4}}^3$ which gives the ratio $D : 2ax = \overline{\frac{5bx}{4}}^3 : \overline{2ax}^3$,

 and

and of consequence, $D = \frac{5bx}{4}$. Thus;

$$\frac{2ax}{1+\frac{5bx}{4}} - \frac{1}{3} \times \overline{\frac{2ax}{1+\frac{5bx}{4}}}\Big|^3 = \text{M nearly}^*.$$

Ex. 2. Suppose $AmL + \frac{1}{2}Am^2L^2 + \frac{1}{6}Am^3L^3 + \frac{1}{24}Am^4L^4$, *&c.* to be an infinite converging series, $(=P.)$

Assume $\left(\frac{AmL}{C+D} =\right) AmL + \frac{1}{2}Am^2L^2 = \frac{AmL}{C} - \frac{AmLD}{C^2}$ . . . . ; then $C = 1$, $D = -\frac{1}{2}mL$, and $\frac{AmL}{1-\frac{1}{2}mL}$ is the first approximating fraction. By division, $\frac{AmL}{1-\frac{1}{2}mL} = AmL + \frac{1}{2}Am^2L^2 + \frac{1}{4}Am^3L^3 + \frac{1}{8}Am^4L^4 + \frac{1}{16}Am^5L^5 +$, *&c.* therefore, by subtraction,

$$P = \frac{AmL}{1-\frac{1}{2}mL} - \frac{1}{12}Am^3L^3 - \frac{1}{12}Am^4L^4 - \frac{13}{240}Am^5L^5,$$ *&c.*

Then, for a second approximating fraction,

Assume $\left(\frac{\frac{1}{12}Am^3L^3}{E+F} =\right) \frac{1}{12}Am^3L^3 + \frac{1}{12}Am^4L^4 = \frac{\frac{1}{12}Am^3L^3}{E} - \frac{\frac{1}{12}Am^3L^3 \times F}{E^2}$ . . . . ; then $E = 1$, $F = -mL$, and therefore,

$$P = \frac{AmL}{1-\frac{1}{2}mL} - \frac{\frac{1}{12}Am^3L^3}{1-mL}, \text{ \&c.}$$

If now we resolve the fraction $\frac{\frac{1}{12}Am^3L^3}{1-mL}$, by division, and

* This approximation is that given by Mr Simpson, without any investigation, in his ESSAYS, p. 47, for determining the equation of a planetary orbit. Very ingenious solutions of Kepler's Problem, have also been given by Sir Isaac Newton, Mr Machin, Dr Keil, Dr Matthew Stewart, *&c.*

proceed

proceed as before, we shall obtain a third approximating fraction, *viz.* $\dfrac{\frac{7}{720} A m^5 L^5}{1 - \frac{14}{6} m L} = \dfrac{\frac{1}{35} A m^5 L^5}{1 - 2 m L}$ nearly. And thus:

$P = \dfrac{A m L}{1 - \frac{1}{2} m L} - \dfrac{\frac{1}{12} A m^3 L^3}{1 - m L} + \dfrac{\frac{1}{35} A m^5 L^5}{1 - 2 m L}$, &c. very nearly.

Proposition XXVI.

85. Supposing n to represent a number: The measures of the ratios $\dfrac{1}{1-n} : 1$, and $1 + n : 1$; or, the logarithms of the quantities $\dfrac{1}{1-n}$, and $1 + n$, will be expressed by the following infinite series, *viz.*

$L, \dfrac{1}{1-n} = M \times \overline{n + \frac{1}{2} n^2 + \frac{1}{3} n^3 + \frac{1}{4} n^4 + \frac{1}{5} n^5, \&c.}$

$L, 1 + n = M \times \overline{n - \frac{1}{2} n^2 + \frac{1}{3} n^3 - \frac{1}{4} n^4 + \frac{1}{5} n^5, \&c.}$

Where M, (which may be expounded by any number taken at pleasure) is that QUANTITY, called by MR COTES, The MODULUS OF THE LOGARITHMIC SYSTEM*.

86. Corol. 1. If we combine together the series for the logarithm of $\dfrac{1}{1-n}$, and that for the logarithm of $1 + n$, we shall obtain a single series, which will converge much faster than either series in this proposition, *viz.*

$L, \dfrac{1+n}{1-n} = 2 M \times \overline{n + \frac{1}{3} n^3 + \frac{1}{5} n^5 + \frac{1}{7} n^7, \&c.}$

From this series, therefore, to derive the logarithm of any quantity, P, we must suppose $P = \dfrac{1+n}{1-n}$, and hence $n = \dfrac{P-1}{P+1}$ a proper fraction, and of consequence, in every case, a converging quantity.

* Harmonia Mensurarum, Prop. I.

87. Corol.

87. Corol. 2. Since the *modulus* M may be interpreted by any numbers taken at pleasure; therefore, DIFFERENT logarithmic numbers may be derived from the same series, (85, 86), by taking for M different numbers. Thus: If the *modulus* $M=1$, the logarithms produced will be those published by the illustrious inventor of logarithms, *viz.* The Honourable John Napier Baron of Merchiston; the logarithm of any quantity P, being, in this case, expounded by

$$2\times\overline{n+\tfrac{1}{3}n^3+\tfrac{1}{5}n^5+,}\ \&c.$$

But, if the *modulus* $M=\frac{1}{2}$, or, if $2M=1$, that is, if we halve all Napier's logarithms, another series will be produced, called NATURAL LOGARITHMS, because, in this case, the logarithm of P, is expressed simply by the series,

$$n+\tfrac{1}{3}n^3+\tfrac{1}{5}n^5,\ \&c.$$

And hence the natural logarithms are properly the ORIGINAL, or RADICAL LOGARITHMIC NUMBERS.

88. Corol. 3. Let Q represent the infinite series producing *natural* logarithms, then, supposing M, *m* to be the *moduli* of ANY two different logarithmic systems, it is evident, that all logarithms, in THESE systems, will be proportional to the *moduli* M, *m*. Therefore, supposing L to represent a logarithm in the system, whereof the *modulus* is M, and *l* a logarithm in the other system whose *modulus* is *m*, we have the ratio $M:m=L:l$. According to Napier's system, where the *modulus* is unity, the logarithm of 10 is the number 2.30258, &c. and according to Napier's improved system, as published by Mr Briggs, the logarithm of 10 is unity; therefore, by substituting these numbers in the above analogy, we have $m=\dfrac{1}{2.30258,\ \&c.}=0.4342944$, &c. the *modulus* of the common system of logarithms. Thus: If we divide all the logarithms in Baron Napier's system, by his logarithm of 10, or, if we multiply them all by the reciprocal of his logarithm of 10, we shall have the logarithms in the common system.

89. Corol. 4. Suppose $M\times\overline{n\pm\frac{1}{2}n^2+\frac{1}{3}n^3\pm\frac{1}{4}n^4,\ \&c.}=L$, then, by reverting this series, (79, Case I) we have $n=\dfrac{L}{m}\mp\dfrac{L^2}{2.M^2}+\dfrac{L^3}{2.3.M^3}\mp\dfrac{L^4}{2.3.4.M^4}$, &c. and of consequence,

(supposing

(supposing $\frac{1}{M} = m$) $1 \mp n = 1 \mp mL + \frac{m^2L^2}{2} \mp \frac{m^3L^3}{2.3} + \frac{m^4L^4}{2.3.4}$, &c. By this series, we may determine the number $1 \mp n$, from the logarithm L, being given.

90. Corol. 5. As the series given in the last corollary, converges very slowly, when L is large, therefore, to remedy this defect, find, by a table of logarithms, the next less logarithm to that given, setting aside the characteristics, and let A be the number corresponding thereto: Then, if z represent the number required, we shall have $\frac{z}{A} = 1 + mL + \frac{1}{2}m^2L^2 + \frac{1}{6}m^3L^3 + \frac{1}{24}m^4L^4$, &c. (89), and therefore, $z = A + AmL + \frac{1}{2}Am^2L^2 + \frac{1}{6}Am^3L^3$, &c. $= A + \frac{AmL}{1 - \frac{1}{2}mL} - \frac{\frac{1}{12}Am^3L^3}{1 - mL} + \frac{\frac{1}{12}Am^4L^4}{1 - 2mL}$ very nearly, (84. *Ex.* 2.) Here, if we take the two first terms of the abbreviated series, as an approximation, near enough for practice, we shall have $z = A + \frac{AmL}{1 - \frac{1}{2}mL} = \frac{\overline{M + \frac{1}{2}L} \times A}{M - \frac{1}{2}L}$ and hence, the ratio $M - \frac{1}{2}L : M + \frac{1}{2}L = A : z$.

Ex. Suppose $z^{365} = 1.06$ to determine the value of z. Here $365 \times L.\ z = L.\ 1.06\ (= L.\ 2 + L.\ 0.53) = 0.0253058652647$ by Mr Sharp's table *, and therefore $L.\ z = 0.000069331137$: By Sherwin's tables, the next lesser logarithm to that of z, is 0.0000434, and Number $A = 1.0001$, and from the first term only of the logarithmic series (86), we get L. 1.0001 ($=$ L. $\frac{10001}{10000}$) $= 0.000043427276821\ (= 2M \times n) = $ L. A. Thus;

$$\begin{array}{l}
L,\ z = 0.00006933113771, \\
L,\ A = 0.00004342727682, \\
\hline
L,\ \frac{z}{A} = 0.00002590386089 = L, \\
\qquad 1.0001 = A \\
\hline
A\,L = 0.0000259064512360085.
\end{array}$$

* See Sherwin's tables, and Sharp's Geometry.

M =

$M = 0.43429448190325$ (58)

$\frac{1}{2}L = 0.00001295193042$

$M - \frac{1}{2}L = 0.43428152997283$

$\frac{AL}{M - \frac{1}{2}L} = 0.0000596535874$

And $z \left(= A + \frac{AL}{M - \frac{1}{2}L}\right) = 1.0001596535874$, by an exceeding easy operation.

91. Corol. 6. If, instead of taking the number corresponding to the next lesser logarithm, as in the last corollary, we take that corresponding to the next greater, and call it B; then proceeding, as in the last corollary, we shall have $z = B - \frac{BmL}{1 + \frac{1}{2}mL} + \frac{\frac{1}{12}Bm^3L^3}{1 + mL} - \frac{\frac{1}{35}Bm^5L^5}{1 + 2mL}$, &c. and taking the two first terms, as an approximation near enough for practice, $z = B - \frac{BmL}{1 + \frac{1}{2}mL} = \frac{\overline{M - \frac{1}{2}L} \times B}{M + \frac{1}{2}L}$, and hence, the ratio $M + \frac{1}{2}L : M - \frac{1}{2}L = B : z$.

Ex. Suppose $z^{365} = 1.06$. Here, I take, from Sherwin's tables, the number corresponding to the logarithm 0.0000434, to six places, *viz.* $1.00016 (= 7.6 \times 0.47 \times 0.28) = B$.

$L, B = 0.00006948155878$, by Sharp's table.

$L, z = 0.00006933113771$

$L, \frac{B}{z} = 0.00000015042102 = L$

$BL = 0.00000001504450873632$

$M + \frac{1}{2}L = 0.43429455711376$

$\frac{BL}{M + \frac{1}{2}L} = 0.0000003464102$,

And therefore, $z \left(= B - \frac{BL}{M + \frac{1}{2}L}\right)$

$= 1.0001596535897$ very nearly, as before.

Note.

Note. The *modulus* for the common logarithms, from Napier's, is,

0.43429448190325182765112891891 6, &c *.

PROPOSITION XXVII.

93. In any series of quantities a, b, c, d, e, f, g, &c. if there be taken all the orders of the differences of the terms, and if A, B, C, D, E, &c. represent the first terms of the several orders of differences: Then, if T represent the first term of the order of differences denoted by n, it will be $T = a - n\,b + n.\dfrac{n-1}{2}\,c - n.\dfrac{n-1}{2}.\dfrac{n-2}{3}\,d$, &c. Where the several coefficients of a, b, c, d, &c. are the UNCIAE of a binomial, raised to a power denoted by n, (10, 78), and the quantity T itself will be affirmative or negative, according as n is an even or odd number.

DEMONSTRATION.

Series, a, b, c, d, e, f, &c.

First differences, $b-a$, $c-b$, $d-c$, $e-d$, $f-e$, &c.

Second, ——— $c-2b+a$, $d-2c+b$, $e-2d+c$, &c.

Third, ——— $d-3c+3b-a$, $e-3d+3c-b$, &c.

Fourth, ——— $e-4d+6c-4b+a$, &c.

&c. &c.

Thus;

$A = -a+b$

$B = a-2b+c$

$C = -a+3b-3c+d$

$D = a-4b+6c-4d+e$

$E = -a+5b-10c+10d-5e+f$,

&c. &c.

* See, on the subject of logarithms, Dr Wallis's Algebra, chap. XII. Sir Isaac Newton's Fluxions, Sharp's Geometry improved, Sherwin's tables, Mr Dodson's Antilogarithmic Canon, Maseres's Trigonometry, &c.

Or,

$$-A = a - b$$
$$B = a - 2b + c$$
$$-C = a - 3b + 3c - d$$
$$D = a - 4b + 6c - 4d + e$$
$$-E = a - 5b + 10c - 10d + 5e - f,$$
&c. &c.

And hence, (10, 78.)

$$a - nb + n.\frac{n-1}{2}c - n.\frac{n-1}{2}.\frac{n-2}{3}d + n.\frac{n-1}{2}.\frac{n-2}{3}.\frac{n-3}{4}e, \text{ \&c.} = \pm T,$$ according as n is even or odd.

Q. E D.

Ex. 1. Required the first term of the 5th order of differences in the series, $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}$, &c.

Here $n = 5$, and therefore, $-T = 1 - 5b + 10c - 10d + 5e - f = \frac{1}{32}$; or, $T = -\frac{1}{32}$.

Ex. 2. Required the first of the 8th order of differences in the series, 1, 3, 9, 27, 81, &c.

Here $n = 8$, and therefore, $T = 1 - 8b + 28c - 56d$, &c. $= 256$.

Ex. 3. Required the first of the 4th order of differences in the series, 1, 8, 27, 64, 125, &c.

Here $n = 4$, and therefore, $T = a - 4b + 6c$, &c. $= 0$.

94. Corol. 1. If, from the equations in this proposition, we derive the quantities, b, c, d, e, f, &c. in terms of a, A, B, C, D, &c. we shall have

$$b = a + A$$
$$c = a + 2A + B$$
$$d = a + 3A + 3B + C$$
$$e = a + 4A + 6B + 4C + D$$
$$f = a + 5A + 10B + 10C + 5D + E,$$
&c. &c.

And hence, it appears, that the value of any term in the series, a, b, c, d, e, f, &c. whose distance, from the first term, is denoted

noted by n, will be $a + nA + n.\frac{n-1}{2}B + n.\frac{n-1}{2}.\frac{n-2}{3}C + n.\frac{n-1}{2}.\frac{n-2}{3}.\frac{n-3}{4}D +$, &c. (10, 78.)

Or, if we take in the first term a, of the series, a, b, c, d, e, f, &c. the nth term of this series, will be

$a + \frac{n-1}{1}A + \frac{n-1}{1}.\frac{n-2}{2}B + \frac{n-1}{1}.\frac{n-2}{2}.\frac{n-3}{3}C + \frac{n-1}{1}.\frac{n-2}{2}.\frac{n-3}{3}.\frac{n-4}{4}D +$, &c. (10, 78.) where the coefficients are the UNCIÆ of the power denoted by $n-1$.

For
$$\begin{aligned}
a &= a \\
b &= a + A \\
c &= a + 2A + B \\
d &= a + 3A + 3B + C \\
e &= a + 4A + 6B + 4C + D \\
f &= a + 5A + 10B + 10C + 5D + E,
\end{aligned}$$
&c. &c.

And thus it appears, that any term of a proposed series, may be *accurately* determined, when the differences of any order are equal among themselves.

Ex. 1. Required the 20th term of the series, 1, 8, 27, 64, 125, &c.?

Here $a = 1$, $A = 7$, $B = 12$, $C = 6$, $D = 0$, and $n = 20$; therefore, $a + 19A + \frac{19.18}{1.2}B + \frac{19.18.17}{1.2.3}C = 8000$, the 20th term required.

Ex. 2. Required the 20th term of the series, 1, 3, 6, 10, 15, 21, &c.?

Here $a = 1$, $A = 2$, $B = 1$, $C = 0$, and $n = 20$; therefore, $a + 19A + 19.9B = 210$.

95. Corol. 2. If we take the equations in the last corollary, (94), expounding the terms of the proposed series, a, b, c, d, e, f,

f, &c. and collect the terms into one sum, we shall have the following equations, *viz.*

$$a = 0 + a$$
$$a + b = 0 + 2a + A$$
$$a + b + c = 0 + 3a + 3A + B$$
$$a + b + c + d = 0 + 4a + 6A + 4B + C$$
$$a + b + c + d + e = 0 + 5a + 10A + 10B + 5C + D$$
$$a + b + c + d + e + f = 0 + 6a + 15A + 20B + 15C + 5D + E,$$

&c. *&c.*

And hence (10, 78.)

$$na + n.\frac{n-1}{2}A + n.\frac{n-1}{2}.\frac{n-2}{3}B + n.\frac{n-1}{2}.\frac{n-2}{3}.\frac{n-3}{4}C + n.\frac{n-1}{2}.\frac{n-2}{3}.\frac{n-3}{4}.\frac{n-4}{5}D +, \&c.$$

will be the sum of n terms of the series, a, b, c, d, e, f, &c. By this theorem, therefore, which was first given by the illustrious Sir Isaac Newton, we can determine the sum of any number of terms in any proposed series.

Thus: The sum of n terms of the series.

$$1, 2, 3, 4, \ldots\ldots n = \overline{n+1} \times \frac{n}{2}.$$

$$1^2, 2^2, 3^2, 4^2, \ldots\ldots n^2 = \overline{n+1} \times \frac{n}{2} \times \frac{2n+1}{3};$$

$$1^3, 2^3, 3^3, 4^3, \ldots\ldots n^3 = \overline{n+1} \times \frac{n}{2} \times \frac{n.\overline{n+1}}{2}$$

&c. *&c.*

In these examples, if we suppose n to be *greater* than *any assignable number*; then,

The LIMIT of the sum of the *infinite* series.

$$1, 2, 3, 4, \ldots\ldots n = n \times \tfrac{1}{2}n.$$

$$1^2, 2^2, 3^2, 4^2, \ldots\ldots, n^2 = n^2 \times \tfrac{1}{3}n.$$

$$1^3, 2^3, 3^3, 4^3, \ldots\ldots n^3 = n^3 \times \tfrac{1}{4}n.$$

&c. *&c.*

Universally,

Universally, $1^m, 2^m, 3^m, 4^m, \ldots n^m = n^m \times \frac{1}{m+1} n$.

And this general example comprehends the whole ARITHMETIC OF INFINITES.

96. Corol. 3. If we ſuppoſe the terms of the ſeries, a, b, c, d, e, f, &c. to be placed, or arranged, in ſuch a way, as that they may be at an UNIT's diſtance from each other, *viz.* $a, . b, . c, . d, . e, . f$, *&c.* Then, if n repreſent the *place* or *diſtance* of any term, x, to be *interpolated*; it will be $x = a + n\text{A} + n.\frac{n-1}{2}\text{B} + n.\frac{n-1}{2}.\frac{n-2}{3}\text{C} +$, *&c.* (94.) And, as this value of x is perfectly accurate, when x is any term of the ſeries, a, b, c, d, e, f, &c. ſo it will be *nearly true*, when x is any intermediate term.

Ex. 1. In the ſeries, $\frac{1}{50}, \frac{1}{51}, \frac{1}{52}, \frac{1}{53}, \frac{1}{54}$, *&c.* let that term be required, which ſtands in the middle, between $\frac{1}{52}$ and $\frac{1}{53}$.

Here $n = 2\frac{1}{2}$, and therefore $x = a + \frac{5}{2}\text{A} + \frac{15}{8}\text{B} + \frac{5}{16}\text{C} - \frac{5}{128}\text{D} = \frac{1}{52\frac{1}{2}}$.

Ex. 2. Having given the logarithmic ſigns of 1° 0′, 1° 1′ 1° 2′, and 1° 3′, to determine the logarithmic ſign of 1° 1′ 40″.

Here (1° 1′ 40″ — 1° = 1′ 40″ =) $1\frac{2}{3} = n$.

	A	B	C
$a = 8.2418553$			
$b = 8.2490332$	71779		
$c = 8.2560943$	70611	-1168	
$d = 8.2630424$	69481	-1130	38

$x = a + \frac{5}{3}\text{A} + \frac{5}{9}\text{B} - \frac{5}{81}\text{C} = 8.2537533$ very nearly.

97. Corol. 4. If we ſuppoſe the terms of the ſeries, a, b, c, d, e, f, &c. to be *anyhow* equidiſtant, and that the quantities, in

in the first order of differences, are small, it is evident, that the quantities, in the successive orders of differences, will continually approximate to equality; and therefore, the quantities, in the last order of differences, will *vanish*, or be equal to nothing: And thus, having any number of terms given, we can, by this proposition, determine any quantity that is wanting in the series, either *accurately* or *nearly*, by supposing the first term T, (93), of the corresponding order (n), of differences, to be equal to nothing.

1. A	$a - b$
2. B	$a - 2b + c$
3. C	$a - 3b + 3c + d$
4. D	$a - 4b + 6c + 4d + e$
5. E	$a - 5b + 10c - 10d + 5e - f$
6. F	$a - 6b + 15c - 20d + 15e - 6f + g$
7. G	$a - 7b + 21c - 35d + 35e - 21f + 7g - h$,
&c.	&c.
4. T	$a - nb + n.\frac{n-1}{2}c - n.\frac{n-1}{2}.\frac{n-2}{3}d + n.\frac{n-1}{2}.\frac{n-2}{3}.\frac{n-3}{4}e$, &c.

Ex. 1. Having given the logarithms of the numbers, 101, 102, 104, and 105, to determine the logarithm of 103.

Here, by supposing $a - 4b + 6c - 4d + e = 0$, we have $c = \frac{4 \times \overline{b+d} - \overline{a+e}}{6} = 2.0128372$ nearly.

$a = 2.0043214$
$b = 2.0086002$
$d = 2.0170333$
$e = 2.0211893$

$4 \times \overline{b+d} = 16.1025340$
$a + e = 4.0255197$
$6)\overline{12.0770233}(=$
$2.0128372.$

Ex. 2. Having given the cube roots of the numbers, 45, 46, 47, 48, and 49, to determine that of 50.

Here, by supposing $a - 5b + 10c - 10d + 5e - f = 0$, we have $f = a - 5b + 10c - 10d + 5e = 3.684033$.

N	Root.
45	$3.556893 = a$
46	$3.583048 = b$
47	$3.608826 = c$
48	$3.634241 = d$
49	$3.659306 = e$

$$a + 10c + 5e = 57.941683$$
$$5b + 10d = 54.257650$$
$$f = 3.684033$$
$$= \sqrt[3]{50} \text{ nearly.}$$

98 Corol. 5. If the terms of the series, a, b, c, d, e, &c. be supposed to be placed at any *unequal* distances, in manner following, *viz.*

$$\begin{array}{ccccccccccc} -x & & x & & x & & x & & x & & x \\ \text{M.} & (m) & \text{N.} & (p) & \text{P.} & (q) & \text{Q.} & (r) & \text{R.} & (s) & \text{S.} \ \ldots \\ & & a & & b & & c & & d & & e \ \ldots \\ & & y & & y & & y & & y & & y \end{array}$$

Let $A = \dfrac{b-a}{p}$

$$\overset{\prime}{A} = \frac{c-b}{q} \quad B = \frac{\overset{\prime}{A} - A}{p+q}$$

$$\overset{\prime\prime}{A} = \frac{d-c}{r} \quad \overset{\prime}{B} = \frac{\overset{\prime\prime}{A} - \overset{\prime}{A}}{q+r} \quad C = \frac{\overset{\prime}{B} - B}{p+q+r}$$

$$\overset{\prime\prime\prime}{A} = \frac{e-d}{s} \quad \overset{\prime\prime}{B} = \frac{A''' - \overset{\prime\prime}{A}}{r+s} \quad \overset{\prime}{C} = \frac{\overset{\prime\prime}{B} - \overset{\prime}{B}}{q+r+s} \quad D = \frac{\overset{\prime}{C} - C}{p+q+r+s}$$

&c. &c. &c. &c.

Then, the term y, corresponding to any place x, will be

$a + A \times \overline{x-m} + B \times \overline{x-m} \times \overline{x-m+p} + C \times \overline{x-m} \times \overline{x-m+p} \times \overline{x-m+p+q}$ +, &c. where m represents the distance (M N) of the first term from any given place (M.) Or, if we suppose the point M to fall upon the point N; then,

$y = a + Ax + Bx \times \overline{x-p} + Cx \times \overline{x-p} \times \overline{x-p-q} + Dx \times \overline{x-p} \times \overline{x-p-q} \times \overline{x-p-q-r}$ +, &c. continued to the same nnmber of terms with the series a, b, c, d, e, &c.

The

The investigation of this general theorem, for interpolating any term of a series, is here omitted, as being more tedious than difficult.

Ex. 1. Given a comet's distance from the sun, on the following days, at 12 at night, to find its distance, December 20, at the same hour.

December 12	. . . distance	$301 = a$
21	. . . .	$620 = b$
24	. . . .	$715 = c$
26	. . . .	$772 = d.$

Here $x\ (= 20 - 12) = 8$, the time after Dec. 12.

$p\ (= 21 - 12) = 9$, $q\ (= 24 - 21) = 3$, and $r\ (= 26 - 24) = 2$.

Also $A = \frac{319}{9} = 35.4444$. $B = -\frac{3.778}{12} = -0.314$, and $D = -\frac{0.3185}{14} = -0.0227$; whence, then $y = 586.9831$ the comet's distance, Dec. 20, at 12 at night.

Ex. 2. Given three distances of the sun from the tropic, and the times of observation. To determine the time of the solstice,

December 20	. . . . . dist.	$39'' = a$
21	. . . .	$6 = b$
24	. . . .	$74 = c$

Here $p\ (= 21 - 20) = 1$, and $q\ (= 24 - 21) = 3$. Also, $A = -33$, and $B = \frac{55\frac{2}{3}}{4} = 13\frac{11}{12}$; whence $y = 39 - 46\frac{11}{12}x + 13\frac{11}{12}x^2 = 0$, y being nothing at the time of the solstice: And therefore, $x = 1.3132$, and $x = 2.058$; (26, 27.) *.

99. Corol. 6. In the series, a, b, c, d, e, f, &c. if we suppose, that there is the *same* ratio between every two adjacent or

* See Sir Isaac Newton's Principia and Quadrature of Curves,—Philosophical Transactions, No 362,—Cotes's Harmonica Mensurarum,—Stirling on Series,—Simpson's Essays, &c. and Emerson's Conic Sections.

contiguous

contiguous terms, then, are the quantities, a, b, c, d, e, f, &c. said to be in *continued proportion*: And, in any series of this kind, all the orders of differences are in the *same* continued proportion, also. For, since by supposition, the ratio $a:b=b:c=c:d=d:e=e:f$, &c. therefore, by conversion and alternation, the ratio $(a:b=)\ a-b:b-c=b-c:c-d=c-d:d-e=d-e:e-f$, &c. (13); and, by continual conversion and alternation, the same will appear with regard to *all* the orders of differences.

100. Corol. 7. In the series of the last corollary, let the ratio $m:1$ be the common ratio of the series; then, if we suppose this ratio to approach nearer to a ratio of equality, than to any other assignable ratio whatsoever, it is evident, that we shall have the first differences, $a-b$, $b-c$, $c-d$, $d-e$, $e-f$, &c. very nearly equal to one another. And therefore, in this case, *Any power or root of any term of the series will have the same ratio to the difference of the same powers or roots of any two adjacent terms, as the term first assumed to such a multiple of the common difference of the terms, as is denoted by the index of the given power or root*, NEARLY. Thus:

$$\left.\begin{array}{ll}\text{The ratio} & a^2:a^2-b^2=a:2\times\overline{a-b}\\ & a^3:a^3-b^3=a:3\times\overline{a-b}\\ & a^4:a^4-b^4=a:4\times\overline{a-b}\\ \text{Universally,} & a^m:a^m-b^m=a:m\times\overline{a-b}\end{array}\right\}\text{nearly.}$$

Also,

$$\left.\begin{array}{ll}\text{The ratio} & a^{\frac{1}{2}}:a^{\frac{1}{2}}-b^{\frac{1}{2}}=a:\tfrac{1}{2}\times\overline{a-b}\\ & a^{\frac{1}{3}}:a^{\frac{1}{3}}-b^{\frac{1}{3}}=a:\tfrac{1}{3}\times\overline{a-b}\\ & a^{\frac{1}{4}}:a^{\frac{1}{4}}-b^{\frac{1}{4}}=a:\tfrac{1}{4}\times\overline{a-b}\\ \text{Universally,} & a^{\frac{m}{n}}:a^{\frac{m}{n}}-b^{\frac{m}{n}}=a:\tfrac{m}{n}\times\overline{a-b}\end{array}\right\}\text{nearly.}$$

And hence we derive a *very easy method* of approximating the roots of all surds or radical quantities.

Ex. 1. Suppose $a=2$, and $b=\sqrt{2}$. (80.)

Then $a^2-b^2:2\times\overline{a-b}=a^2:a$, which gives $b=\frac{3}{2}$ nearly.

M Also,

Alſo, $a^2 - b^2 : 2 \times \overline{a - b} = b^2 : b$, which gives $b = \frac{4}{3}$ nearly. As the former approximating quantity $\frac{3}{2}$ is greater than the root, and the latter nearly as much leſs, therefore, by taking the *intermediate* quantity, we obtain $\frac{17}{12}$ a near approximation. And by repeating this operation, the root may be determined to any degree of accuracy.

Ex. 2. Suppoſe $a = 3$, and $b = \sqrt{12}$. (80.)

Here $b^2 - a^2 : 2 \times \overline{b - a} = a^2 : a$, which gives $b = \frac{7}{2}$.

Alſo $b^2 - a^2 : 2 \times \overline{b - a} = b^2 : b$, which gives $b = \frac{24}{7}$.

And by taking the *intermediate* quantity, we obtain the fraction $\frac{97}{28}$ a near approximation.

Ex. 3. Suppoſe $a = 1$, and $b = \sqrt[365]{1.06}$ (90, 91.)

Here $b^{365} - a^{365} : 365 \times \overline{b - a} = a^{365} : a$, from which $b = \frac{365.06}{365}$.

Alſo $b^{365} - a^{365} : 365 \times \overline{b - a} = b^{365} : b$, from which $b = \frac{386.9}{386.84}$.

And by taking the *intermediate* quantity, we obtain 1.0001597 a very near approximation to the root.

Proposition XXVIII.

101. If we aſſume any quantity at pleaſure, and therefrom derive a ſeries of terms, increaſing or decreaſing, *by equal differences*; the terms of the ſeries are ſaid to be in CONTINUED ARITHMETICAL PROGRESSION: And one half the number of terms in the ſeries, will have the ſame *ratio* to unity, as the ſum of all the terms to the ſum of the two extreme terms.

Demonstration.

Let a, v, repreſent the two extreme terms in a continued arithmetical progreſſion, whereof d is the *common* difference of the terms, n the number, and s the ſum of all the terms: Then, ac-

according as we take a or v for the leading quantity, the series will be expressed by the following terms, *viz.*

$$a,\ a \pm d,\ a \pm 2d,\ a \pm 3d,\ a \pm 4d\ .\ a \pm \overline{n-1} \times d\,(=v),$$

$$v,\ v \mp d,\ v \mp 2d,\ v \mp 3d,\ v \mp 4d\ .\ v \mp \overline{n-1} \times d\,(=a.)$$

And these series being perfectly equivalent (taking the terms of the latter backwards, or from right to left), it is manifest, that the two taken together, will be double any one of them; and therefore, $\overline{a+v} + \overline{a+v} + \overline{a+v} + \overline{a+v}$, &c. to n terms, $=\overline{a+v} \times n$ (5) $= 2s$, from whence the ratio $\frac{n}{2} : 1 = s : a+v$, (16.)

Q. E. D.

102. Corol. 1. Since, then $a \pm \overline{n-1} \times d = v$, and $v \mp \overline{n-1} \times d = a$, (101), therefore, $\overline{n-1} \times d = a \backsim v$, and hence the *ratio* $1 : n-1 = d : a \backsim v$, (16.) Thus: In any continued arithmetical progression, unity hath the same *ratio* to the excess of the number of terms above unity, as the common difference of the terms to the difference of the two extreme terms.

103. Corol. 2. Hence, in a continued arithmetical progression, if *any three* of the five quantities, a, v, d, n, and s, are supposed to be GIVEN, the remaining *two* may be determined, as in the following table, where the expressions belong to an increasing progression, and may be applied also to a decreasing progression, by making the quantities a and v, change places every where.

Cases	Data	Quæsita	DETERMINATION.
I.	$a, d, n,$	v	$= a + \overline{n-1} \times d$, (101, 102)
		s	$= n \times \overline{a + d \times \frac{n-1}{2}}$

II.	a, d, v	n	$=\frac{v-a}{d}+1$
		s	$=\frac{v+a}{2}\times\overline{\frac{v-a}{d}+1}$
III.	a, d, s	n	$=\frac{-a+\sqrt{\overline{a\times\overline{a-d}+d\times\overline{2s+d}}}}{d}+\frac{1}{2}$
		v	$=\sqrt{\overline{a\times\overline{a-d}+d\times\overline{2s+d}}}-\frac{1}{2}d$
IV.	a, v, s	n	$=\frac{2s}{v+a}=2s\times\frac{1}{v+a}$ (101.)
		d	$=\frac{\overline{v+a}\times\overline{v-a}}{2s-\overline{v+a}}$
V.	a, n, s	v	$=\frac{2s}{n}-a=\overline{2s-na}\times\frac{1}{n}$ (101.)
		d	$=\frac{1}{n-1}\times\frac{2s-2na}{n}=\frac{1}{n-1}\times\overline{\frac{2s}{n}-2a}$
VI.	a, n, v	d	$=\frac{v-a}{n-1}=\overline{v-a}\times\frac{1}{n-1}$ (102.)
		s	$=\frac{n}{2}\times\overline{a+v}=n\times\frac{a+v}{2}$ (101.)

VII.	d, v, n	a	$=v-\overline{n-1}\times d$ (101.)
		s	$=n\times\overline{v-d\times\frac{n+1}{2}}$
VIII.	d, v, s	a	$=\frac{1}{2}d\pm\sqrt{v\times\overline{v+d}+d\times\overline{2s-\frac{1}{4}d}}$
		n	$=\frac{v\mp\sqrt{v\times\overline{v+d}+d\times\overline{2s-\frac{1}{4}d}}}{d}+\frac{1}{2}$
IX.	d, n, s	a	$=\frac{s}{n}-\frac{1}{2}d\times\overline{n-1}$ (101, 102.)
		v	$=\frac{s}{n}+\frac{1}{2}d\times\overline{n-1}$ (101, 102.)
X.	n, v, s	a	$=\frac{2s}{n}-v=\overline{2s-nv}\times\frac{1}{n}$ (101.)
		d	$=\frac{2v}{n}+\frac{2}{n}\times\frac{v-s}{n-1}$.

104. Corol. 3. Since univerſally $s=\frac{n}{2}\times\overline{a+v}$ (101); therefore, if $a=0$, then $s=\frac{n}{2}\times v$. Thus: the ſum of a continued arithmetical progreſſion, whereof one of the extreme terms is a cypher, (0), is equal to the other extreme term repeated as often as there are units in half the number of terms in the ſeries.

105. Corol. 4. From Caſe I. in Corol. 2. it appears that $s=n\times\overline{a+d\times\frac{n-1}{2}}$; therefore, if we ſuppoſe $a=2$, and $d=2$ alſo, then $s=n^2+n$: And, in this caſe, s is called a PRO-

NIC

NIC NUMBER. Thus: *pronic* numbers are produced by adding together *even* numbers in a continued arithmetical progreſſion.

Continued Arithmetical Progreſſion, 2, 4, 6, 8, 10, 12, 14, 16, &c.
Pronic Numbers Numbers, 2, 6, 12, 20, 30, 42, 56, 72, &c.

Moreover, ſince $n^2 + n = s$, therefore, (26), $n = \frac{\sqrt{4s+1}-1}{2}$ the pronic root of the number denoted by s: And hence, *if we increaſe the quadruple of any propoſed number by unity; then, half the exceſs of the ſquare root of this ſum, above unity, will be the pronic root of the propoſed number.*

106. Corol. 5. Further, ſince univerſally $s = n \times \overline{a + d \times \frac{n-1}{2}}$; therefore, if we ſuppoſe $a = 1$, and $d = 2$, then $s = n^2$: And, thus all ſquare numbers may be conſidered as originating from the addition of odd numbers in a continued arithmetical progreſſion.

Continued Arithmetical Progreſſion, 1, 3, 5, 7, 9, 11, 13, 15, 17, &c.
Square Numbers, 1, 4, 9, 16, 25, 36, 49, 64, 81, &c.

107. Corol. 6. From the laſt corollary may be derived a general theorem, for determining *how* many odd numbers muſt be added together to produce a *given power of that number.* Let the given number be n, the exponent of its power m, and the firſt term of the ſeries a; then, ſince in a ſeries of odd numbers, the difference of the terms is 2, (106), and number of terms n, by ſuppoſition; therefore, $(s =) n^m = n \times \overline{a + \overline{n-1}}$, and of conſequence $a = n^{m-1} - \overline{n-1}$: And thus it appears, how powers of all dimenſions may be obtained by the SUMMING up of *odd* numbers.

Thus: Suppoſing $m = 3$, and expounding n by the numbers 2, 3, 4, 5, 6, 7, &c.

$2^3 = 3 + 5$
$3^3 = 7 + 9 + 11$
$4^3 = 13 + 15 + 17 + 19$
$5^3 = 21 + 23 + 25 + 27 + 19$
$6^3 = 31 + 33 + 35 + 37 + 39 + 41,$
&c. &c.

Again:

Again: Suppofing $m = 4$, and expounding n as before.

$2^4 = 7 + 9$

$3^4 = 25 + 27 + 29$

$4^4 = 61 + 63 + 65 + 67$

$5^4 = 121 + 123 + 125 + 127 + 129$

$6^4 = 211 + 213 + 215 + 217 + 219 + 221$,

&c. &c.

And we may proceed in the fame manner for any other powers.

108. Corol. 7. If we take the common arithmetical progreffion of natural numbers, proceeding regularly from unity, and terminating in any *fquare number*, the terms of this progreffion may be fo difpofed within the cells of a geometrical fquare, that the fums of each row, taken diagonally, laterally and vertically, fhall be the fame; and fuch arithmetical fquares are called MAGIC SQUARES. Here, as the fide (or root) of the terminating fquare number, may be either odd, evenly-even, or unevenly-even, fo, in the conftruction of magic fquares, there may be three different cafes.

109. Cafe 1. When the fide of the fquare number is odd. Range all the numbers in their natural order within the cells of a geometrical fquare; this fquare is called *the generator*, or *generating fquare*. The diagonal and the lateral middle column of the generator, give the vertical middle column and the diagonal of the magic fquare; at the extremities of the diagonal of the magic fquare, defcend and afcend vertically, one term, increafing and decreafing; continue upon diagonals, defcending and afcending vertically at their extremities as before; and the remaining cells will be filled up with terms contiguous to thofe, at oppofite extremeties of adjoining columns vertically and laterally, continuing upon diagonals as before.

GENERATOR.
Square of 3.

1	2	3
4	5	6
7	8	9

MAGIC SQUARE.
Side 3.

8	1	6
3	5	7
4	9	2

GENE-

Generator.
Square of 5.

1	2	3	4	5
6	7	8	9	10
11	12	13	14	15
16	17	18	19	20
21	22	23	24	25

Magic Square.
Side 5.

17	24	1	8	15
23	5	7	14	16
4	6	13	20	22
10	12	19	21	3
11	18	25	2	9

Magic Square.
Side 7.

30	39	48	1	10	19	28
38	47	7	9	18	27	29
46	6	8	17	20	35	37
5	14	16	24	33	36	45
13	15	24	33	42	44	4
21	23	32	41	43	3	12
22	31	40	49	2	11	20

Magic Square.
Side 9.

47	58	69	80	1	12	23	34	45
57	68	79	9	11	22	33	44	46
67	78	8	10	21	32	43	54	56
77	7	18	20	31	42	53	55	66
6	17	19	30	41	52	63	65	76
16	27	29	40	51	62	64	75	5
26	28	39	50	61	72	74	4	15
36	38	49	60	71	73	3	14	25
37	48	59	70	81	2	13	24	35

110. Case 2. When the side of the square number is evenly-even, or divisible by 4 †.

Here we must first construct the magic square, whose side is 4, the least evenly-even number. In order to which, having formed the generator as in case 1, let the terms on both diagonals remain, and exchange the other exterior terms cross-ways with those that are opposite to them, and the square will be completed.

Generator.
Square of 4.

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

Magic Square.
Side 4.

1	15	14	4
12	6	7	9
8	10	11	5
13	3	2	16

† See Memoires de l'Academie Royale des Sciences pour l'année 1750.

Having

Having thus obtained the magic ſquare of 4, all other magic ſquares, whoſe ſides are evenly-even, may be derived by ranging all the terms of the progreſſion into two rows; the terms in the upper row called *ſmall* numbers, being in the natural order from unity; and the terms of the lower row called the *complements* of the ſmall numbers, being in a retrograde order, ſeparate both rows, by drawing a line through them at every eight number; then theſe parcels of fixteen figures muſt be formed into ſquares, according to the method uſed above in conſtructing the magic ſquare of 4, and the magic ſquare will be completed, by joining theſe ſubſidiary ſquares, ſo as to form a ſquare any how.

The Conſtruction of the MAGIC SQUARE, *whoſe ſide is* 8.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
64	63	62	61	60	59	58	57	56	55	54	53	52	51	50	49	48	47	46	45	44	43	42	41	40	39	38	37	36	35	34	33

GENERATORS.

1	2	3	4
5	6	7	8
57	58	59	60
61	62	63	64

9	10	11	12
13	14	15	16
49	50	51	52
53	54	55	56

17	18	19	20
21	22	23	24
41	42	43	44
45	46	47	48

25	26	27	28
29	30	31	32
33	34	35	36
37	38	39	40

SUBSIDIARY SQUARES.

1	63	62	4
60	6	7	57
8	58	59	5
61	3	2	64

9	55	54	12
52	14	15	49
16	50	51	13
53	11	10	56

17	47	46	20
44	22	23	41
24	42	43	21
45	19	18	48

25	39	38	28
36	30	31	33
32	34	35	29
37	72	26	40

MAGIC SQUARE.
Side 8.

1	63	62	4	9	55	54	12
60	6	7	57	52	14	15	49
8	58	59	5	16	50	51	13
61	3	2	64	53	11	10	56
17	47	46	20	25	39	38	28
44	22	23	41	36	30	31	33
24	42	43	21	32	34	35	29
45	19	18	48	37	27	26	40

MAGIC SQUARE.

Side 12.

1	143	142	4	9	135	134	12	17	127	126	20
140	6	7	137	132	14	15	129	124	22	23	121
8	138	139	5	16	130	131	13	24	122	123	21
141	3	2	144	133	11	10	136	125	19	18	128
25	119	118	28	33	111	110	36	41	105	102	44
116	30	31	113	108	38	39	105	100	46	47	97
32	114	115	29	40	100	107	37	48	96	99	45
117	27	26	120	109	35	34	112	101	43	42	104
49	95	94	52	57	87	86	60	65	79	78	68
92	54	55	89	84	62	63	81	76	70	71	73
56	90	91	53	64	82	83	61	72	74	75	79
93	51	50	96	85	59	58	88	77	67	66	80

111. Case 3. When the side of the square number is unevenly-even.

Here we must first construct the magic square whose side is 6, the smallest unevenly-even number. In order to which, let all the numbers in the regular progression be distributed into two rows, as in the last case, in manner following:

1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18,
36, 35, 34, 33, 32, 31, 30, 29, 28, 27, 26, 25, 24, 23, 22, 21, 20, 19,

Then the square of 4 being the greatest evenly-even square that can be inscribed in the square of 6, we may take the eight last of the small numbers, with their complements for filling up this square, and the ten preceding numbers, with their complements, remain to fill up the twenty cells of the *border*; or, we may take the first eight numbers with their complements, or those in the middle to fill the square, while the other extreme numbers, or those in the middle and extremes remain for the border. Here then we shall take the first ten small numbers, with their complements for the border, *viz.*

1, 2, 3, 4, 5, 6, 7, 8, 9, 10,
36, 35, 34, 33, 32, 31, 30, 29, 28, 27,

*x				*z	*y
	11	25	24	14	*x
	22	16	17	12	*z
*y	18	20	21	16	
*z	23	13	12	26	
	*x	*y	*z		

The internal part of the ſquare being conſtructed, as in the laſt caſe (110), and each ſmall number with its complement, making up the ſelf-ſame ſum, therefore any ſmall number that is placed in the border, muſt have its complement oppoſite to it; and any ſmall number, placed at one extremity of either diagonal, muſt have its complement at the other extremity of the ſame diagonal. If now we mark ten cells in the border, each with an aſteriſm (*) for the ten ſmall numbers, then the three ſmall numbers to be placed in each row of the border, making up the ſame ſum, may be expreſſed in general terms, by x, y, and z, and 1, 2, 3, &c. to 10 terms, being equal to 55, (101) therefore, ($3x + 3y + 4z$ or) $3 \times \overline{x+y} + 4z = 55$ which indeterminate equation being reſolved by Prop. XIX. and its corrollaries (40—48) we obtain four different general methods for placing the ſmall numbers in the proper cells of the border, *viz.*

$$\begin{array}{rcccc|c} x+y = & 17, & 13, & 9, & 5 & 1 \\ z = & 1, & 4, & 7, & 10 & 13 \\ \hline x+y+z = & 18, & 17, & 16, & 15 & \end{array}$$

and as the ſmall numbers x, y, z, to be placed in the border, may be derived from any one of theſe equations, we ſhall therefore take the laſt values of $x+y$ and z, theſe being in the *leaſt* numbers. Wherefore, having placed 10, the value of z, in the cell of the upper row belonging to it, ſince the two ſmall numbers x, y, to be placed in the extreme cells of this row taken together, amount to 5, we may therefore take either the numbers 1 and 4,

					z	y
x	*1				*10	*4
						*2
						*9
y	*6					
z	*8					
		*3	*5	*7		
		x	y	z		

or 2 and 3; let us take then the numbers 1 and 4, and expunge these with the number 10, out of the series of small numbers

~~1~~, 2, 3, ~~4~~, 5, 6, 7, 8, 9, ~~10~~.

Since the two small numbers x, z, to be placed in the right hand row of the border, taken together, amount to 11, we may take for them either 2 and 9, or 3 and 8; let us take then the numbers 2 and 9, and expunge them out of the series of small numbers.

~~1~~, ~~2~~, 3, ~~4~~, 5, 6, 7, 8, ~~9~~, ~~10~~,

And since the small numbers y, z, to be placed in the left hand row of the border, amount to 14, we have therefore 6 and 8 to be placed in these cells; wherefore, expunging these numbers out of the series of small numbers

~~1~~, ~~2~~, 3, ~~4~~, 5, ~~6~~, 7, ~~8~~, ~~9~~, ~~10~~,

we have remaining for the three cells of the lower border, the small numbers 3, 5, 7. The border is then completed, by writing opposite to each small number, x, y, z, its respective complement.

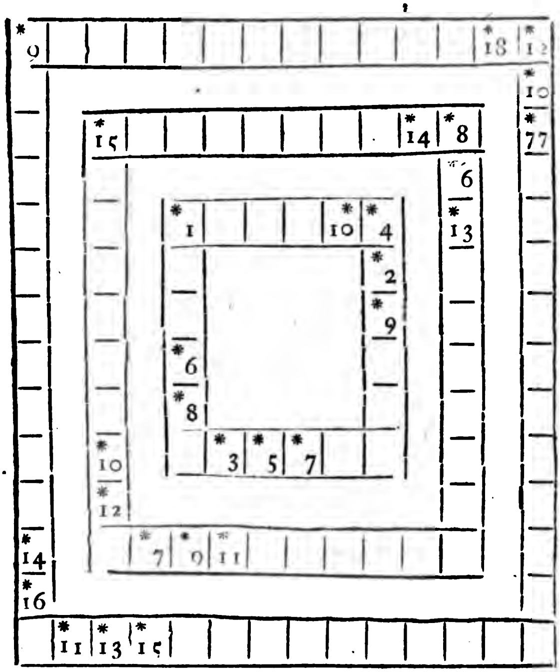
Magic Square.
Side 6.

*1	34	32	30	*10	*4
35	11	25	24	14	*2
28	22	16	17	19	*9
*6	18	20	21	16	31
*8	23	13	12	6	29
33	*3	*5	*7	27	36

112.

112. Having thus obtained the magic ſquare, whoſe ſide is 6, we may, in the ſelf-ſame manner, conſtruct the magic ſquare of any other number, whoſe ſquare root is unevenly-even. Alſo, in conſtructing magic ſquares, whoſe ſides are numbers unevenly-even, we may proceed in this manner, *viz.*

Diſtribute all the numbers of the progreſſion into two rows, as in the laſt caſe, ſeparating them into parcels of ſixteen figures, by drawing a line at every eight number, reckoning from the furthermoſt extremity; complete theſe ſquares, and join them into one, as in the laſt caſe; and thus we ſhall have the internal part of the ſquare. To obtain the numbers to be placed in the border, from the number denoting the ſide of the ſquare to be conſtructed, take the number 6, and add the remainder to the numbers in the extreme cells of the ſquare of 6, marked with an aſteriſm (*) placing the ſums in the correſponding cells of the ſquare to be conſtructed, and their



complements in the oppoſite extreme cells. Laſtly, take the remaining ſmall numbers, and diſtribute them into two rows, placing

cing the greater under the leffer as before; feparate the numbers in thefe two rows into four divifions, and place all the numbers of the fame divifion in the fame row of the border, which will be completed by writing the complements of the fmall numbers in the cells oppofite to them.

The Conftruction of the MAGIC SQUARE, *whofe fide is* 10.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
100	99	98	97	96	95	94	93	92	91	90	89	88	87	86	85	84	83	82	81	80	79	78	77	76

26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50
75	74	73	72	71	70	69	68	67	66	65	64	63	62	61	60	59	58	57	56	55	54	53	52	51

Small

Small numbers produced by adding 4 (=10—6) to the numbers in the asterism cells of the square of 6.

5, 6, 7, 8, 9, 10, 11, 12, 13, 14

Remaining small numbers disposed according to the rule, and separated into four divisions.

1	2	3	4
18	17	16	15

MAGIC SQUARE.

Side 10.

*5	94	92	90	85	98	1	18*	14	*8
95	19	81	80	22	27	73	72	30	*6
88	78	24	25	75	70	32	33	67	*13
99	26	76	77	23	34	68	69	31	2
84	79	21	20	82	71	29	28	74	17
15	35	65	64	38	43	57	56	46	86
4	62	40	41	59	54	48	49	51	97
*10	42	60	61	39	50	52	53	47	91
*12	63	37	36	66	55	45	44	58	89
93	*7	*9	*11	16	3	100	83	87	96

The Construction of the MAGIC SQUARE, *whose side is* 14.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26
196	195	194	193	192	191	190	189	188	187	186	185	184	183	182	181	180	179	178	177	176	175	174	173	172	171

27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50
170	169	168	167	166	165	164	163	162	161	160	159	158	157	156	155	154	153	152	151	150	149	148	147

51	52	53	54	55	56	57	58	59	60	61	62	63	64	65	66	67	68	69	70	71	72	73	74
146	145	144	143	142	141	140	139	138	137	136	135	134	133	132	131	130	129	128	127	126	125	124	123

75	76	77	78	79	80	81	82	83	84	85	86	87	88	89	90	91	92	93	94	95	96	97	98
122	121	120	119	118	117	116	115	114	113	112	111	110	109	108	107	106	105	104	103	102	101	100	99

Small

Small numbers in the asterism cells. 9, 10, 11, 12, 13, 14, 15, 16, 17, 18.

Small numbers separated into four divisions.

1	[illegible]	3	4	5	6	7	8
26	25	24	23	22	21	20	19

MAGIC SQUARE.

Side 14.

*9	186	184	182	192	175	191	176	1	26	2	25	*18	*12
187	27	169	168	30	35	161	160	38	43	153	152	46	*10
180	166	32	33	163	158	40	41	155	150	48	49	147	*17
194	34	164	165	31	42	156	157	39	50	148	149	47	3
173	167	29	28	170	159	37	36	162	151	45	44	154	24
193	51	145	144	54	59	137	136	62	67	129	128	70	4
174	142	56	57	139	134	64	65	131	126	72	73	123	23
7	58	140	141	55	66	132	133	63	74	124	125	71	190
20	143	53	52	146	135	61	60	138	127	69	68	130	177
8	75	121	120	78	83	113	112	86	91	105	104	94	189
19	118	80	81	115	110	88	89	107	102	96	97	99	178
*14	82	116	117	79	90	108	109	87	98	100	101	95	183
*16	119	77	76	122	111	85	84	114	103	93	92	106	181
185	*11	*13	*15	5	22	6	21	196	171	195	172	179	188

113 Corol. 8. The general increaſing ſeries of this propoſition $a, a + d, a + 2d, a + 3d, a + 4d, \ldots . a + \overline{n-1} \times d$ (101) becomes $1, 1 + d, 1 + 2d, 1 + 3d, 1 + 4d, \ldots .$ $1 + \overline{n-1} + d$ ſuppoſing a equal to unity; and combining together the terms of this latter ſeries, according to the *number of terms*, we obtain the following ſeries, expreſſing all *combinatory or polygonal numbers*, viz. $1, 1 + \overline{1 + d}, 1 + \overline{1 + d} + \overline{1 + 2d}, 1 + \overline{1 + d}, + \overline{1 + 2d}, + \overline{1 + 3d}; 1 + \overline{1 + d}, + \overline{1 + 2d}, + \overline{1 + 3d}, + \overline{1 + 4d}, \ldots . 1 + \frac{\overline{n-1} \times d}{2} \times n$ (103, 105).

Thus,

Thus expounding the common difference d by 0, 1, 2, 3, 4, 5, &c. we have the following ſeries of arithmetical progreſſionals, and of polygonal numbers, *viz.*

	ARITHMETICAL PROGRESSIONS.	POLYGONAL NUMBERS.
$d=0$	1, 1, 1, 1, 1, 1, 1, 1, 1, &c. Units.	1, 2, 3, 4, 5, 6, 7, 8, 9, &c. Laterals.
$d=1$	1, 2, 3, 4, 5, 6, 7, 8, 9, &c. Laterals.	1, 3, 6, 10, 15, 21, 28, 36, 45, &c. Triangulars.
$d=2$	1, 3, 5, 7, 9, 11, 13, 15, 17, &c.	1, 4, 9, 16, 25, 36, 49, 64, 81, &c. Quadrangulars.
$d=3$	1, 4, 7, 10, 13, 16, 19, 22, 25, &c.	1, 5, 12, 22, 35, 51, 70, 92, 117, &c. Pentangulars.
$d=4$	1, 5, 9, 13, 17, 21, 25, 29, 33, &c.	1, 6, 15, 28, 45, 66, 91, 120, 153, &c. Hexagonals.
$d=5$	1, 6, 11, 16, 21, 26, 31, 36, 41, &c.	1, 7, 18, 34, 55, 81, 112, 148, 209, &c. Heptagonals.
&c.	&c.	&c.

And as the laſtterm of the combinatory ſeries $1+\frac{\overline{\overline{n-1}\times d}}{2}\times n$ repreſents, in general, any polygonal number; therefore, expounding d as before, by 0, 1, 2, 3, 4, 5, &c.

$1 . n .$ Lateral number.	$1 . n . \frac{n+1}{2}$ Triangular number.
$1 . n . n$ Quadrangular or ſquare number.	$1 . n . \frac{3n-1}{2}$ Pentagonal number.
$1 . n . \overline{2n-1}$ Hexagonalnumber.	$1 . n . \frac{5n-3}{2}$ Heptagonal number.

&c.

Where a point is uſed inſtead of the ſign of multiplication.

114. Taking the series of units, and combining its terms continually, as also the terms of every succeeding series, we obtain all the orders of *triangular figurate numbers*, Thus:

Figurates of the 1st order, 1, 1, 1, 1, 1, 1, 1, 1, 1, &c. Units.
———— 2d order, 1, 2, 3, 4, 5, 6, 7, 8, 9. Laterals.
———— 3d order, 1, 3, 6, 10, 15, 21, 28, 36, 45, &c. Triangulars.
———— 4th order, 1, 4, 10, 20, 35, 56, 84, 120, 125, &c. 1st Triangular Pyramidals.
———— 5th order, 1, 5, 15, 35, 70, 126, 210, 330, 495, &c. 2d Triangular Pyramidals.
———— 6th order, 1, 6, 21, 56, 126, 252, 462, 792, 1287, &c. 3d Triangular Pyramidals.
&c. &c. &c.

The numbers in these series taken vertically, being evidently the same with the *unciæ* of the corresponding terms of a binomial, raised to all integral powers (9); therefore, if we suppose n to represent the side of a figurate number of any order, or which is the same, the exponent of any order, then 1, $1.n$, $1.n.\frac{n+1}{2}$, $1.n.\frac{n+1}{2}.\frac{n+2}{3}$, $1.n.\frac{n+1}{2}.\frac{n+2}{3}.\frac{n+3}{4}$, &c. will represent the first, second, third, fourth, fifth, &c. figurative numbers of that order; or supposing n as before (113.) to represent the number of terms, the terms of this series, after the first, will represent all triangular figurate numbers, Thus:

$1.n$ Lateral number.

$1.n.\frac{n+1}{2}$ Triangular number.

$1.n.\frac{n+1}{2}.\frac{n+2}{3}$ 1st Triangular Pyramidal number.

$1.n.\frac{n+1}{2}.\frac{n+2}{3}.\frac{n+3}{4}$ 2d Triangular Pyramidal number.

$1.n.\frac{n+1}{2}.\frac{n+2}{3}.\frac{n+3}{4}.\frac{n+4}{5}$ 3d Triangular Pyramidal number.

&c. &c.

The

The same expressions may be universally derived from the general polygonal number $1+\frac{\overline{n-1}\times d}{2}\times n$ by the second corollary of the last Proposition (95). Thus, $\overline{1+\frac{\overline{n-1}\times d}{3}\times n}.\frac{n+1}{2}$ or, (taking $d=1$) $n.\frac{n+1}{2}.\frac{n+2}{3}$ represents any 1st Triangular Pyramidal; and hence all the other orders of Triangular Pyramidals as above.

115. These expressions give us also all the *permutations* and all the *different combinations* of quantities. The continual product of the series of laterals n, $n-1$, $n-2$, $n-3$, $n-4$, *&c.* continued to the units in n, gives directly all the permutations of any number (n) of quantities; and if any quantity occur twice, thrice, four times, *&c.* then $\frac{n.\overline{n-1}.\overline{n-2}.\overline{n-3}.\overline{n-4}.\&c}{2.1}$, $\frac{n.\overline{n-1}.\overline{n-2}.\overline{n-3}.\overline{n-4}.\&c.}{3.2.1}$, $\frac{n.\overline{n-1}.\overline{n-2}.\overline{n-3}.\overline{n-4}.\&c}{4.3.2.1}$, *&c.* will be the respective permutations. And, as for the combinations of quantities, these, when the quantities are taken by two's, three's, four's, *&c.* go on according to the numbers in the series of triangulars, 1st triangular pyramidals, 2d triangular pyramidals, *&c.* therefore, supposing n to represent the number of quantities, and r the exponent of the combination, then shall $\overline{n-r+1}.\frac{\overline{n-r+2}}{2}.\frac{\overline{n-r+3}}{3}.\frac{\overline{n-r+4}}{4}.$ *&c.* to $\frac{n}{r}$ represent the combinations of r quantities in n: That is,

$\overline{n-r+1}.\frac{n}{r}$ combinations of two quantities in n.

$\overline{n-r+1}.\frac{\overline{n-r+2}}{2}.\frac{n}{r}$ three quantities in n.

$\overline{n-r+1}.\frac{\overline{n-r+2}}{2}.\frac{\overline{n-r+3}}{3}.\frac{n}{r}$ four quantities in n.

&c. *&c.*

116.

116. The sums of all the combinations found by taking 0, 1, 2, 3, 4; *&c.* quantities out of any number (n) of things proposed, is called the *election* of quantities, and is expressed universally by $\dot{2}$, or excluding 0 by $\dot{2}-1$, or excluding both 0, 1 by $\dot{2}-\overline{n+1}$.

117. Corol. 9. By taking the reciprocals of the terms of an arithmetical progression, we obtain a series of quantities, in continued HARMONICAL, OR MUSICAL PROPORTION; and by taking the difference of every two contiguous terms of the harmonical series, the LAW of this series will be evident by inspection, Thus:

Arithmetical series. $a,\ a\pm d,\ a\pm 2d,\ a\pm 3d,\ a\pm 4d,\ a\pm 5d, \ldots \ldots a\pm\overline{n-2}\times d,\ a\pm\overline{n-1}\times d$

Harmonical series. $\frac{1}{a},\ \frac{1}{a\pm d},\ \frac{1}{a\pm 2d},\ \frac{1}{a\pm 3d},\ \frac{1}{a\pm 4d},\ \frac{1}{a\pm 5d}, \ldots \ldots \frac{1}{a\pm\overline{n-2}\times d},\ \frac{1}{a\pm\overline{n-1}\times d},$

Difference of the terms of Harmonical series. $\left\{ \frac{1}{a}\cdot\frac{d}{a\pm d},\ \frac{1}{a\pm d}\cdot\frac{d}{a\pm 2d},\ \frac{1}{a\pm 2d}\cdot\frac{d}{a\pm 3d},\ \frac{1}{a\pm 3d}\cdot\frac{d}{a\pm 4d},\ \frac{1}{a\pm 4d}\cdot\frac{1}{a\pm 5d}, \ldots \ldots \frac{1}{a\pm\overline{n-2}\times d}\cdot\frac{d}{a\pm\overline{n-1}\times d}. \right.$

From the differential series it appears directly, that in the harmonical progression, there is always the same *ratio* between the two differences of any three contiguous terms, as there is between the two extremes of the said three contiguous terms; also, that there is the same *ratio* between the product of the two first terms of the harmonical series, and the product of any two other contiguous terms, as there is between the two differences of said contiguous terms. And hence,

Supposing a series of harmonical quantities represented by } A, B, C, D, E, F, ... V
And the series of differences by M, N, P, Q, R, ... X
Then A=B±M, B=C±N, C=D±P, D=E±Q, E=F±R, *&c.*

Therefore,

Therefore, the ratio $M : N = B \pm 1\, M : C$

—— $M : P = B \pm 2\, M : D$

—— $M : Q = B \pm 3\, M : E$

—— $M : R = B \pm 4\, M : F$

&c. &c.

And universally, the ratio $M : X = B \pm \overline{n - 2}\,.\, M : V$

Thus, in any series of quantities in harmonical or musical progression, there is the same *ratio* between the difference of the two first terms, and the difference of any two other contiguous terms, as there is between the difference or sum of the second term, and such a multiple of the difference of the two first terms, as is denoted by the number of intermediate terms, and the last term of the series.

118. All the terms of a continued harmonical series may be derived from the two first terms only. Thus A being the first term, B the second, and M their difference; since $A = \frac{A \times B}{B}$, and $B\ (= \frac{A \times B}{A}) = \frac{A \times B}{B \pm M}$ (117) therefore the harmonical progression will be represented by the following series of terms, viz.

$$\frac{A \times B}{B}\ (= A),\ \frac{A \times B}{B \pm M},\ \frac{A \times B}{B \pm 2\, M},\ \frac{A \times B}{B \pm 3\, M},\ \frac{A \times B}{B \pm 4\, M},\ \ldots\ \frac{A \times B}{B \pm \overline{n - 2}\,.\, M}$$

And from this series it is obvious, that a harmonical progression, consisting of *any whatever* number of terms, cannot always be derived from *any* two numbers or quantities A, B taken at pleasure.

119. As three quantities, A, B, C, are in continued harmonical or musical progression, when the ratio $A \backsim B : B \backsim C = A : C$ (117): so, four quantities, A, B, C, D, are *said* to be in harmonical or musical proportion; when the ratio $A \backsim B : C \backsim D = A : D$, and three quantities, A, B, C, are said to be in *contra-harmonic* proportion, when the ratio $A \backsim B : B \backsim C = C : A$. Thus,

These three numbers, 10, 15, 30, are in continued harmonical proportion. These four numbers, 10, 16, 24, 60, are in harmonical

monical proportion; and these three 6, 5, 3, or these 12, 10, 4, are in *contra-harmonic* proportion.

120. Corol. 10. From the arithmetical progression of odd numbers, 1, 3, 5, 7, 9, 11, 13, 15, 17, 19, *&c.* which may be continued for ever, we have shewn how all powers of the numbers in the natural progression, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, *&c.* may be produced by addition only (107). Further, this same progression of odd numbers, indefinitely extended, contains all Prime Numbers, except the number 2; and the law of the progression is such, that *between every two multiples of any number N, as they stand in their natural order in the series, there constantly intervene N—1 numbers, which are not multiples of N.* Hence all prime numbers, except the number 2, may be obtained from the progression of odd numbers, by the following operation, which Dr Horsly † takes to be that called

The Sieve of Eratosthenes.

Let the series of odd numbers be continued indefinitely,
1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31, 33, 35, 37, 39, 41, 43, 45, 47, 49, 51, 53, 55, 57, 59, 61, 63, 65, 67, 69, 71, 73, 75, 77, 79, 81, 83, 85, 87, 89, 91, 93, 95, 97, 99, 101, 103, 105, 107, 109, 111, 113, 115, 117, 119, 121, 123, 125, 127, 129, 131, 133, 135, 137, 139, 141, 143, 145, 147, 149, 151, 153, 155, 157, 159, 161, 163, 165, *&c. &c.*

Then, (1.) Count all the terms of the series following the number 3 by threes, expunging every third number; and thus, all the multiples of 3 will be exterminated. ‡ (2.) The first uncancelled number in the series after 3 being 5, expunge its square (25), and count all the terms of the series, which follow the square of 5 by fives, expunging every fifth number, if not expunged before; and thus, all the multsples of 5 will be exterminated, which were not at first exterminated among the multiples of 3. (3.) The next uncancelled number in the series after 5 being 7, expunge its square (49), and count all the terms of the series which follow the square of 7, by sevens, expunging every seventh number if not expunged before; and thus, all the multiples of 7 will be exterminated, which were not before exterminated among the multiples of 3 and 5. (4.) The next uncancelled number in the series after 7 being 11, expunge its square (121), and count all the terms of the series which follow the square of 11 by elevens, expunging

† Philosophical Transactions, vol. 62. p. 327.

‡ The numbers expunged have a point placed over them.

expunging every eleventh number, if not expunged before; and thus, all the multiples of 11 will be exterminated, which were not before exterminated among the multiples of 3, 5 and 7. (5.) Lastly, continue these expunctions 'till the first uncancelled number that appears, next to that whose multiples have been last expunged, is such, that its square is greater than the last and greatest number, to which the series is extended. The numbers which then remain uncancelled, are all the prime numbers, except the number 2, which occur in the natural progression of numbers, from 1 to the limit of the series.

Thus, the following numbers are all primes, *viz.*

1, [2], 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, 101, 103, 107, 109, 113, 127, 131, 137, 139, 149, 151, 157, 163, *&c.*

121. The natural progression 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, *&c.* indefinitely extended, comprehending *all numbers*, prime, composite, *&c.* let n represent each term of the series of natural numbers, indefinitely extended; then, $n \times 6 - 1$ and $n \times 6 + 1$ will constitute a series comprehending all prime numbers, except the numbers 2 and 3.

122. Corol. 11. General theorems, resolving all the cases of Simple Interest, relative to annuities, pensions, rents, *&c.* in arrear or reversion, may be derived by summing the terms of an arithmetical progression. Thus, supposing p to represent any principal sum lent out for the time t, and that r is the *ratio* of the rate, or the interest of 1 *l.* for one year, then the interest of any principal p, being manifestly in the compound ratio of the principal p, time t, and rate r; therefore, supposing that s is the amount of the principal p, and its interest trp for the time t, all the varieties in this case are resolved from the general equation $\overline{1+rt} \times p = s$, as in the following table.

p, r, t	$s = \overline{1+rt} \times p$
s, r, t	$p = \dfrac{s}{1+rt}$
s, r, p	$t = \dfrac{s-p}{rp}$
s, p, t	$r = \dfrac{s-p}{tp}$

123. Suppose now a to represent any annuity, pension, rent, &c. in *arrear*; then, since 1 *l.* hath the same absolute *ratio* to its rate (r) as any annuity, &c. (a) to its rate ($a\,r$); therefore,

$a \times \overline{1}$ 1st year's amount.

$a \times \overline{1+r}$ 2d year's amount.

$a \times \overline{1+2\,r}$ 3d year's amount.

$a \times \overline{1+3\,r}$ 4th year's amount.

&c. &c.

Universally, $a \times \overline{1+\overline{t-1} \times r}$ amount for time, t.

But the number of terms of this series being t, the sum s is $t\,a \times \overline{1+\frac{t-1}{2}r}$ (101, 103); and from this equation, all the varieties in this case are resolved as in the following table.

$a,\ t,\ r$	$s = t\,a \times \overline{1+\frac{t-1}{2}r}$
$s,\ t,\ r$	$a = \frac{s}{t} \times \frac{1}{1+\frac{t-1}{2}r}$
$s,\ t,\ a$	$r = \frac{\overline{\frac{s}{t}-a} \times 2}{\overline{t-1} \times a}$
$s,\ r,\ a$	$t = \sqrt{\frac{2\,s}{a\,r}+\overline{\frac{1}{r}-\frac{1}{2}}^{2}}\ \ \overline{\frac{1}{r}-\frac{1}{2}}$

124. But if a represents any annuity, pension, rent. &c. in *reversion*, then, since the amount of 1 *l.* (122) for any time, ($1+t\,r$) hath the same absolute *ratio* to 1 *l.* as the amount (123) of any annuity, &c. ($t\,a \times \overline{1+\frac{t-1}{2}r}$) to the present value ($s$) of said annuity, &c. therefore all the varieties in this case, will be resolved as in the following table.

The Construction of the MAGIC SQUARE, *whose side is* 14.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26
196	195	194	193	192	191	190	189	188	187	186	185	184	183	182	181	180	179	178	177	176	175	174	173	172	171

27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50
170	169	168	167	166	165	164	163	162	161	160	159	158	157	156	155	154	153	152	151	150	149	148	147

51	52	53	54	55	56	57	58	59	60	61	62	63	64	65	66	67	68	69	70	71	72	73	74
146	145	144	143	142	141	140	139	138	137	136	135	134	133	132	131	130	129	128	127	126	125	124	123

75	76	77	78	79	80	81	82	83	84	85	86	87	88	89	90	91	92	93	94	95	96	97	98
122	121	120	119	118	117	116	115	114	113	112	111	110	109	108	107	106	105	104	103	102	101	100	99

Small numbers in the asterism cells. 9, 10, 11, 12, 13, 14, 15, 16, 17, 18.

Small numbers separated into four divisions.

1	2	3	4	5	6	7	8
26	25	24	23	22	21	20	19

Magic Square.
Side 14.

*9	186	184	182	192	175	191	176	1	26	2	25	*18	*12
187	27	169	168	30	35	161	160	38	43	153	152	46	*10
180	166	32	33	163	158	40	41	155	150	48	49	147	*17
194	34	164	165	31	42	156	157	39	50	148	149	47	3
173	167	29	28	170	159	37	36	162	151	45	44	154	24
193	51	145	144	54	59	137	136	62	67	129	128	70	4
174	142	56	57	139	134	64	65	131	126	72	73	123	23
7	58	140	141	55	66	132	133	63	74	124	125	71	190
20	143	53	52	146	135	61	60	138	127	69	68	130	177
8	75	121	120	78	83	113	112	86	91	105	104	94	189
19	118	80	81	115	110	88	89	107	102	96	97	99	178
*14	82	116	117	79	90	108	109	87	98	100	101	95	183
*16	119	77	76	122	111	85	84	114	103	93	92	106	181
185	*11	*13	*15	5	22	6	21	196	171	195	172	179	188

113 Corol. 8. The general increasing series of this proposition a, $a + d$, $a + 2d$, $a + 3d$, $a + 4d$, ... $a + \overline{n-1} \times d$ (101) becomes 1, $1 + d$, $1 + 2d$, $1 + 3d$, $1 + 4d$, ... $1 + \overline{n-1} + d$ supposing a equal to unity; and combining together the terms of this latter series, according to the *number of terms*, we obtain the following series, expressing all *combinatory or polygonal numbers*, viz. 1, $\overline{1 + 1 + d}$, $1 + \overline{1 + d} + \overline{1 + 2d}$, $1 + \overline{1 + d}, + \overline{1 + 2d}, + \overline{1 + 3d}$; $1 + \overline{1 + d}, + \overline{1 + 2d}$, $+ \overline{1 + 3d}, + \overline{1 + 4d}$, ... $1 + \frac{\overline{n-1} \times d}{2} \times n$ (103, 105).

Thus,

Thus expounding the common difference d by 0, 1, 2, 3, 4, 5, &c. we have the following series of arithmetical progressionals, and of polygonal numbers, *viz.*

	ARITHMETICAL PROGRESSIONS.	POLYGONAL NUMBERS.
$d=0$	1, 1, 1, 1, 1, 1, 1, 1, 1, &c. Units.	1, 2, 3, 4, 5, 6, 7, 8, 9, &c. Laterals.
$d=1$	1, 2, 3, 4, 5, 6, 7, 8, 9, &c. Laterals.	1, 3, 6, 10, 15, 21, 28, 36, 45, &c. Triangulars.
$d=2$	1, 3, 5, 7, 9, 11, 13, 15, 17, &c.	1, 4, 9, 16, 25, 36, 49, 64, 81, &c. Quadrangulars.
$d=3$	1, 4, 7, 10, 13, 16, 19, 22, 25, &c.	1, 5, 12, 22, 35, 51, 70, 92, 117, &c. Pentangulars.
$d=4$	1, 5, 9, 13, 17, 21, 25, 29, 33, &c.	1, 6, 15, 28, 45, 66, 91, 120, 153, &c. Hexagonals.
$d=5$	1, 6, 11, 16, 21, 26, 31, 36, 41, &c.	1, 7, 18, 34, 55, 81, 112, 148, 209, &c. Heptagonals.
&c.	&c.	&c.

And as the lastterm of the combinatory series $1 + \frac{\overline{n-1} \times d}{2} \times n$ represents, in general, any polygonal number; therefore, expounding d as before, by 0, 1, 2, 3, 4, 5, &c.

$1 . n .$ Lateral number.	$1 . n . \frac{n+1}{2}$ Triangular number.
$1 . n . n$ Quadrangular or square number.	$1 . n . \frac{3n-1}{2}$ Pentagonal number.
$1 . n . \overline{2n-1}$ Hexagonalnumber.	$1 . n . \frac{5n-3}{2}$ Heptagonal number.
&c.	

Where a point is used instead of the sign of multiplication.

114. Taking the series of units, and combining its terms continually, as also the terms of every succeeding series, we obtain all the orders of *triangular figurate numbers*, Thus:

Figurates of the 1st order,	1, 1, 1, 1, 1, 1, 1, 1, 1, &c. Units.
——— 2d order,	1, 2, 3, 4, 5, 6, 7, 8, 9. Laterals.
——— 3d order,	1, 3, 6, 10, 15, 21, 28, 36, 45, &c. Triangulars.
——— 4th order,	1, 4, 10, 20, 35, 56, 84, 120, 125, &c. 1st Triangular Pyramidals.
——— 5th order,	1, 5, 15, 35, 70, 126, 210, 330, 495, &c. 2d Triangular Pyramidals.
——— 6th order,	1, 6, 21, 56, 126, 252, 462, 792, 1287, &c. 3d Triangular Pyramidals.
&c.	&c. &c.

The numbers in these series taken vertically, being evidently the same with the *unciæ* of the corresponding terms of a binomial, raised to all integral powers (9); therefore, if we suppose n to represent the side of a figurate number of any order, or which is the same, the exponent of any order, then 1, $1.n$, $1.n.\frac{n+1}{2}$, $1.n.\frac{n+1}{2}.\frac{n+2}{3}$, $1.n.\frac{n+1}{2}.\frac{n+2}{3}.\frac{n+3}{4}$, &c. will represent the first, second, third, fourth, fifth, &c. figurative numbers of that order; or supposing n as before (113.) to represent the number of terms, the terms of this series, after the first, will represent all triangular figurate numbers, Thus:

$1.n$ Lateral number.

$1.n.\frac{n+1}{2}.\frac{n+2}{3}$ 1st Triangular Pyramidal number.

$1.n.\frac{n+1}{2}.\frac{n+2}{3}.\frac{n+3}{4}.\frac{n+4}{5}$ 3d Triangular Pyramidal number.

&c.

$1.n.\frac{n+1}{2}$ Triangular number.

$1.n.\frac{n+1}{2}.\frac{n+2}{3}.\frac{n+3}{4}$ 2d Triangular Pyramidal number.

&c.

The

VI.	a,n,v	r	$= \overline{\frac{v}{a}}\Big\vert^{\frac{1}{n-1}} \quad - 1.$
		s	$= \dfrac{v - a}{\overline{\frac{v}{a}}\Big\vert^{\frac{1}{n-1}} - 1.}$
VII.	r,v,n	a	$= \dfrac{v}{r^{n-1}} = v\, r^{-\overline{n-1}}.$
		s	$= \dfrac{v}{r^{n-1}} \times \dfrac{r^n - 1}{r^n - 1} = \dfrac{v}{r^{n-1}} \times Q = Q \times v\, r^{-\overline{n-1}}.$
VIII.	r,v,s	a	$= \overline{r-1} \times \overline{rv-s}.$
		n	$= \dfrac{L, v - L, \overline{r-1} \times \overline{rv-s}}{L, r} + 1.$
IX.	r,n,s	a	$= s \times \dfrac{r-1}{r^n-1} = s \times \dfrac{1}{Q} = \dfrac{s}{Q}.$
		s	$= r^n - 1\ s \times \dfrac{r-1}{r^n-1} = r^{n-1} \times \dfrac{s}{Q}.$
X.	n,v,s	a	Equation $s \times \overline{a}\vert^{\frac{1}{n-1}} - \overline{a}\vert^{\frac{n}{n-1}} = \overline{v}\vert^{\frac{1}{n-1}} \times \overline{s-v}.$
		r	Equation $\dfrac{s}{s-v}\, r^{n-1} - r^n = \dfrac{v}{s-v}.$

130. Corol. 3. As the quantities a, v may change places every-where (127, 129;) therefore, in any continued geometrical progression, since the ratio $1 : r = s-v : s-a$ if we always expound r (which is generally called *the common ratio of the series*) by the quotient of the greater by the lesser contiguous term, we shall have universally the ratio $1 : r-1 = s-v : v \backsim a$ (13). Thus: in any continued geometrical progression, there is the same *ratio* between unity, and the excess of the common ratio above unity, as there is between the excess of the sum of all the terms of the series above the greatest or least term, and the difference of the two extreme terms.

131. Corol. 4. From the genesis of a continued geometrical progression, it appears, that one of the extreme terms, *repeated* as often as there are units in the other extreme term, is always equal

equal to any immediate term, at any distance from one of the extremes, *repeated* as often as there are units in the intermediate term, equi-distant from the other extreme; and if the number of terms be odd, then one extreme term, repeated as often as there are units in the other extreme term, is always equal to the square or second power of the middle term.

132. Corol. 5. Between the adjacent terms of any continued geometrical progression, there always obtain *equal ratios;* and the ratio of the two extreme terms is such a multiple of the common ratio of the series, as is denoted by the number of equal ratios, from one extreme term to the other.

Thus, in the series a, ar, ar^2, ar^3, ar^4, ar^5, . . . ar^{n-2}, ar^{n-1}, the ratios $a : ar$, $ar : ar^2$, $ar^2 : ar^3$, $ar^3 : ar^4$, $ar^4 : ar^5$ . . . $ar^{n-2} : ar^{n-1}$ are all equal; as being each equivalent to the ratio $1 : r$, which is therefore the common ratio of the series; and the ratio $a : ar^{n-1} = 1 : r^{n-1} = \overline{n-1} \times \overline{1 : r}$ (5).

133. Corol. 6. By stating the equal ratios between the adjacent terms of a continued geometrical progression, as in the last corollary (132); it is obvious, that $s-a$ is the sum of all the antecedent, and $s-v$ that of all the consequent terms of the series (127); and therefore, in any continued geometrical progression, the ratio between any two adjacent terms, taken in succession, is always equal to the ratio between the sum of all the antecedent, and that of all the consequent terms of the series (15).

134. Corol. 7. If out of any series of quantities in continued geometrical progression, there be taken any series of equi-distant terms, that series will also be in continued geometrical progression. Thus, let the proposed geometrical progression be,

$$a,\ ar,\ ar^2,\ ar^3,\ ar^4,\ ar^5,\ ar^6,\ ar^7,\ ar^8,\ ar^9,\ ar^{10},\ \&c.$$

$$\text{Then, the series} \begin{cases} a,\ ar^2,\ ar^4,\ ar^6,\ ar^8,\ ar^{10},\ \&c. \\ a,\ ar^3,\ ar^6,\ ar^9,\ \&c. \\ ar,\ ar^3,\ ar^5,\ ar^7,\ ar^9,\ \&c. \\ \&c. \qquad\qquad \&c. \end{cases}$$

are all in continued geometrical progression (127).

135. Corol 8. Supposing the leading quantity a of the general series a, ar, ar^2, ar^3, ar^4, ar^5, ar^6, &c. equal to unity; it is obvious, that the terms 1, r, r^2, r^3, r^4, r^5, r^6, &c. are in the same continued geometrical progression as before; and therefore, by the last corollary (134) the following series are all in continued geometrical progression, *viz.*

$$1,\ r^2,\ r^4,\ r^6,\ r^8,\ r^{10},\ \&c.$$
$$1,\ r^3,\ r^6,\ r^9,\ r^{12},\ r^{15},\ \&c.$$
$$1,\ r^4,\ r^8,\ r^{12},\ r^{16},\ r^{20},\ \&c.$$

Universally, $\quad 1,\ r^m,\ r^{2m},\ r^{3m},\ r^{4m},\ r^{5m},\ \&c.$ Thus:

Thus: in any continued geometrical progression, if the first term be any power, then all the other terms of the series are similar powers, or powers under the same exponent.

136. Corol. 9. The series a, ar, ar^2, ar^3, ar^4, ... ar^{n-1}, and a, ar^{-1}, ar^{-2}, ar^{-3}, ar^{-4}, . . ar^{-n-1}, may be considered as derived, by repeating the terms of a series of equal quantities, a, a, a, a, a . . . . . a as often as there are units in the corresponding terms of the continued geometrical progressions 1, r, r^2, r^3, r^4, ... r^{n-1} and 1, r^{-1}, r^{-2}, r^{-3}, r^{-4}, ... r^{-n-1} : and hence it appears, that the common ratio of any continued geometrical progression is not changed, by taking any equimultiples, integral or fractional, of all the terms of the progression (132.) The most simple continued geometrical progressions, therefore, are such, whereof the first or initial term is unity. Thus, supposing m to represent any number, the following are the most simple continued geometrical progressions, in the ratio of 1 to r^m or r^{-m}.

$$1,\ r^m,\ r^{2m},\ r^{3m},\ r^{4m},\ r^{5m},\ r^{6m},\ \&c.$$
$$1,\ r^{-m},\ r^{-2m},\ r^{-3m},\ r^{-4m},\ r^{-5m},\ r^{-6m},\ \&c.$$
or (7, 8)
$$r^0,\ r^m,\ r^{2m},\ r^{3m},\ r^{4m},\ r^{5m},\ r^{6m},\ \&c.$$
$$r^0,\ r^{-m},\ r^{-2m},\ r^{-3m},\ r^{-4m},\ r^{-5m},\ r^{-6m},\ \&c.$$

In these series, the exponents of the terms are evidently in continued arithmetical progression (101); and the *ratios* of the terms of the geometrical progression, to the first term, are denoted by these exponents; or, the exponents are LOGARITHMS of the corresponding terms of the geometrical progression.

137. Corol. 10. Since, in a continued decreasing geometrical progression, the ratio $1 : r-1 = s-a : a-v$ (130); therefore, supposing v to vanish, or the decreasing series to have *no last term*, and in this case, the ratio $1 : r-1 = s-a : a$ which gives $s = \frac{ra}{r-1} = ra \times \frac{1}{1-r}$ (16): and this expression $ra \times \frac{1}{r-1}$ is therefore the LIMIT of the sum, supposing the series continued *indefinitely*; that is, supposing the series continued to a number of terms, *greater* than any assignable number. For, if we resolve the expression $ra \times \frac{1}{r-1}$ by division (7); or by Sir Isaac Newton's binomial theorem (78), we shall obtain directly the proposed decreasing series, *viz.* $a + \frac{a}{r} + \frac{a}{r^2} + \frac{a}{r^3} + \frac{a}{r^4}$, &c. or

$\times \overline{1 + \frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3} + \frac{1}{r^4}}$ &c. which series may be continued for ever; and as the sum of this series varies continually, according as the number (n) of terms vary, so the varying sum constantly approaches nearer and nearer, to the determinate quantity $ra \times \frac{1}{n-1}$, and may, by increasing the number (n) of terms, be brought nearer to this quantity than by any assignable difference; but can never, by any accession of terms, be made actually to reach, much less to pass beyond it; for the determinate quantity $ra \times \frac{1}{r-1}$ arises only from the supposition of the series having no last term, or from that of its being continued *indefinitely*. Here all the varieties which can take place are exhibited as in the following table, *viz.*

a, r	$s = ra \times \frac{1}{r-1}$
a, s	$r = s \times \frac{1}{s-a}$
r, s	$a = s \times 1 - \frac{1}{r}$

138. Corol. 11. Let a, b be the two initial terms of a series decreasing *indefinitely*; then, since $\frac{a}{b} = r$ (130) the LIMIT o the sum is therefore $\frac{a^2}{a-b} = a^2 \times \frac{1}{a-b}$: and hence the series itself is $a + b + \frac{b^2}{a} + \frac{b^3}{a^2} + \frac{b^4}{a^3} +$, &c. *indefinitely*; or, $a \times \overline{1 + \frac{b}{a} + \frac{b^2}{a^2} + \frac{b^3}{a^3} + \frac{b^4}{a^4}}$ +, &c. *indefinitely*; with which series, any decreasing geometrical progression may be compared, and the LIMIT of the sum of consequence determined.

139. Corol. 12. Supposing n, m to represent numbers, then, from the terms of the continued geometrical progression, n^m, n^{m-1}, n^{m-2}, n^{m-3}, n^{m-4}, ... n the COMPOSITION of quantities may be derived. Thus, the composition of m quantities in n is n^m, that of $m-1$ quantities in n is n^{m-1}, that of $m-2$

 quantities

quantities in n is n^{m-2}, and so on; and hence, from the sum of this continued geometrical progression, (130) we shall have this expression $n \times \frac{n^m - 1}{n - 1}$ or (taking $m = n$) $n \times \frac{n^n - 1}{n - 1}$ which denotes all the several compositions or changes there may be of n quantities, combined all possible ways, or taking them by one's, two's, three's, &c. to m or n. Thus, all the compositions or changes of 4 quantities is ($n^n =$) $4^4 = 256$, taking each time 4; and therefore, $n \times \frac{n^n - 1}{n - 1} = 340$ is the number of all the compositions or changes of 4 quantities, combined all possible ways: Thus also, all the compositions or changes of 24 quantities, is ($n^n =$) $24^{24} = 1333735776850284124449$, by taking each time 24; and therefore, the number of all possible compositions or changes, in this case, is $n \times \frac{n^n - 1}{n - 1} = 1391724288887252999425128493402200$, and so many ways may the 24 letters of the alphabet be put together among themselves.

140. Corol. 13. According to EUCLID *, a number which is equal to all its *aliquot* parts taken together, is a PERFECT NUMBER. And *if, from unity, there be taken a series of numbers in the ratio of 1 to 2, (136) continued to such a number of terms, that the sum of all the terms is a prime number, the product of this sum, by the last term of the series, will be a* perfect number †. Suppose, therefore, $s = 1 + 2 + 4 + 8 \ldots + 2^{n-1}$, where s is a prime number; then $2s = 2 + 4 + 8 + 16 \ldots + 2^{n-1} + 2^n$; and therefore, $(2s - s =)\ s = 2^n - 1$: whence, according to EUCLID, $2^{n-1} \times 2^n - 1$, or $s \times 2^{n-1}$ is a perfect number, $2^n - 1$, or s, being a prime number. Since then $s = 2^n - 1$, therefore $s + 1 = 2^n = 2 \times 2^{n-1}$, (6), and $\frac{s+1}{2^{n-1}} = \frac{2}{1}$ (Ax. x); whence, by PROP. XIX. (40—44).

$$s + 1 = 2,\ 4,\ 6,\ 8,\ \&c.\ \text{(com. Diff. 2.)}$$
$$2^{n-1} = 1,\ 2,\ 3,\ 4,\ \&c.\ \text{(Ditto 1.)}$$

Taking then, $\begin{matrix} s + 1 = 2,\ 4 \\ 2^{n-1} = 1,\ 2 \end{matrix} \left\{ \begin{matrix} s = 1,\ 3 \\ 2^{n-1} = 1,\ 2 \end{matrix} \right.$

And from these two initial values of s and 2^{n-1} (which give the first two perfect numbers 1 and 6) all the other values may be obtained, by continuing the series therefrom, (42); considering on-

* Definition XXII. Book VII. † Euclid, Book IX. Prop. XXXVI.

ly that s must be a prime number, $n-1$ an even number, and of consequence n an odd number. Thus,

$s = 1$	$\overline{1+2\times2^{1}-1}$	$\overline{1+2\times2^{2}-1}$	$\overline{1+2\times2^{4}-1}$	$\overline{1+2\times2^{6}-1}$	$\overline{1+2\times2^{8}-1}$
$2^{n-1}=1$	2	2^{2-1}	2^{4-1}	2^{6-1}	2^{8-1}
$n = 1$	2	3	5	7	9

$\overline{1+2\times2^{10}-1}$	$\overline{1+2\times2^{12}-1}$	$\overline{1+2\times2^{14}-1}$	$\overline{1+2\times2^{16}-1}$	&c.
2^{16-1}	2^{12-1}	2^{14-1}	2^{16-1}	&c.
11	13	15	17	&c.

This series, which may be continued for ever, will assign all perfect numbers; and that in a more general and determinate manner, than any other series which hath hitherto been given.

141. Corol. 14. Two numbers, A, B, are said to be AMIABLE, when all the aliquot parts of A taken together make up the number B, and those of B taken together make up the number A; so that amiable numbers may be considered as *interchangeable* perfect numbers. And universally, if 2^n, (where n is an affirmative integer), be taken such, that $3\times\overline{2^n-1}$, $6\times\overline{2^n-1}$ and $18\times\overline{2^{2n}-1}$ be prime numbers; then, $2^{n+1}\times 18\,.\,2^{2n}-1$ shall be an amiable number, and $2^{n+1}\times 3\,.\,2^n-1\times 6\,.\,2^n-1$ its partner; or, supposing $2^n=Q$, and that $3\,\overline{Q-1}$, $6\,Q-1$ and $18\,\overline{Q^2-1}$ are prime numbers, then $2\,Q\times 18\,\overline{Q^2-1}$ and $2\,Q\times 3\,\overline{Q-1}\times 6\,\overline{Q-1}$, are numbers amiable to one another. Here expounding n by 1, 3, 6, the first three pairs of amiable numbers are 284, and 220, 18416 and 17296, 9437056, and 9363584. And this general theorem, according to SCHOOTEN * was first given by the celebrated DES CARTES.

142. Corol. 15. General theorems, resolving all the cases of COMPOUND INTEREST, relative to annuities, pensions, rents, &c. in arrear, reversion, or perpetuity, may be derived by summing the terms of a continued geometrical progression. Suppose p to represent any principal sum lent out for the time t, r the rate, or the interest of 1 *l.* for one year, and R $(=1+r)$ the amount of 1 *l.* and its interest for one year (called here the ratio of the rate); then, since the ratio $1:R=R:R^2=R^2:R^3=R^3:R^4$, &c. (136) therefore R^t will universally be the amount of 1 *l.* for any time denoted by t: whence, since the amounts are in the same *ratio* with the principal sums, therefore, supposing s to be the amount of the principal p, and its interest $p\,R^t$ for the time t,

* *Sectiones Miscellaniæ*, p. 423.

all the varieties in this case are resolved from the general equation $p\,R^t = s$, as in the following table:

p, R, t	$p\,R^t = s$ or $L, p + t \times L, R = L, s$
s, R, t	$p = \frac{s}{R^t}$ or $L, p = L, s - t \times L, R$
s, R, p	$t = \frac{L, s - L, p}{L, R}$
s, p, t	$R = \sqrt[t]{\frac{s}{p}}$, or $L, R = \frac{L, s - L, p}{t}$

143. Suppose now a to represent any annuity, pension, rent, *&c.* in *arrear;* then, since the last year's annuity, *&c.* bears no interest, as tarying out no *time*, the first year's annuity, *&c.* will therefore universally be represented by $a\,R^{-1}$: whence, deriving a series by this proposition (127), from the quantity a in the ratio $1 : R$ we shall have $a + a\,R + a\,R^2 + a\,R^3 \ldots\ldots + a\,R^t = s$, or $s = a \times \frac{R^t - 1}{R - 1}$ (129, 130;) and from this general equation, all the varieties in this case are resolved as in the following table:

a, t, R	$s = a \times \frac{R^t - 1}{r}$ (142).
s, t, R	$a = r\,s \times \frac{1}{R^t - 1}$
s, t, a	$\frac{s}{a} R - R^t = \frac{s}{a} - 1$
s, R, a	$t = \frac{L, \frac{rs}{a} + 1}{L, R}$

144. But, if a represents an annuity, pension, rent, *&c.* in *reversion*, then, since the ratio $1 : \frac{1}{R} = a : \frac{a}{R}$ the present worth of the annuity, *&c.* a at the end of one year will therefore be $\frac{a}{R}$: whence, deriving a series by this proposition (127) from the quantity

quantity $\frac{a}{R}$ in the ratio $1 : \frac{1}{R}$ we shall have $\frac{a}{R} + \frac{a}{R^2} + \frac{a}{R^3} + \frac{a}{R^4}$ $\ldots\ldots + \frac{a}{R^t} = s$ or $s = a \times \frac{1 - \frac{1}{R^t}}{R - 1}$ (129, 130); and from this equation, all the varieties in this case are resolved as in the following table:

a, t, R	$s = a \times \frac{1 - \frac{1}{R^t}}{r}$ (142)
s, t, R	$a = rs \times \frac{1}{1 - \frac{1}{R^t}}$
s, t, a	$\overline{1 + \frac{a}{s}} \times R^t - R^{t+1} = \frac{a}{s}$
s, R, a	$t = \frac{L, a - L, \overline{a - rs}}{L, R}$

145. If we suppose the progression $\frac{a}{R} + \frac{a}{R^2} + \frac{a}{R^3} + \frac{a}{R^4}$ $\ldots\ldots + \frac{a}{R^t}$ (144.) to have *no last term*; then, on this supposition, a representing a PERPETUITY, or the yearly rent of a FREE-HOLD ESTATE, all the varieties in the buying and selling of estates in fee-simple, may be resolved from the equation $s = a \times \frac{1}{R - 1}$ (137) as in the following table:

a, R	$s = a \times \frac{1}{r}$
s, R	$a = rs$
s, a	$R = 1 + a \times \frac{1}{s}$

146. And if we suppose N to represent the number of years purchase a is worth, then, Na ($= s$) being the worth itself of a, therfore, instead of a substituting its value (rs) from the fee-simple equation (145), and we shall have $Nr = 1$. Thus, the number

number (N) of years purchase a fee-simple, is worth at the rate R of compound interest is $\frac{1}{r}$; and the rate (R) of compound interest, at which the purchase-money of a fee-simple is valued for N years purchase is $1+\frac{1}{N}$.

147. Lastly, since $\frac{1}{r}\left(=\frac{1}{R-1}\right)=\frac{1}{R}+\frac{1}{R^2}+\frac{1}{R^3}+\frac{1}{R^4}+$, &c. therefore, the sum of the values of 1 *l.* for the *successive years* of the life of a person of a given age, would be expressed by this series, were there no contingency in the case. And to allow for contingency, a table of the probabilities of life must be used; and hence may be computed the following tables:

TABLE

TABLE I. *by Mr Thomas Simpson, shewing the value of an annuity of* 1 *l.* *upon one* LIFE. *

Age.	Years purchase, at 5 *per cent.*	Ditto, at 4 *per cent.*	Ditto, at 3 *per cent.*	Age.	Years purchase, at 5 *per cent.*	Ditto, at 4 *per cent.*	Ditto, at 3 *per cent.*
6	14.1	16.2	18.8	41	10.2	11.4	13.0
7	14.2	16.3	18.9	42	10.1	11.2	12.8
8	14.3	16.4	19.0	43	10.0	11.1	12.6
9	14.3	16.4	19.0	44	9.9	11.0	12.5
10	14.3	16.4	19.0	45	9.8	10.8	12.3
11	14.3	16.4	19.0	46	9.7	10.7	12.1
12	14.2	16.3	18.9	47	9.5	10.5	11.9
13	14.1	16.2	18.7	48	9.4	10.4	11.8
14	14.0	16.0	18.5	49	9.3	10.2	11.6
15	13.9	15.8	18.3	50	9.2	10.1	11.4
16	13.7	15.6	18.1	51	9.0	9.9	11.2
17	13.5	15.4	17.9	52	8.9	9.8	11.0
18	13.4	15.2	17.6	53	8.8	9.6	10.7
19	13.2	15.0	17.4	54	8.6	9.4	10.5
20	13.0	14.8	17.2	55	8.5	9.3	10.3
21	12.9	14.7	17.0	56	8.4	9.1	10.1
22	12.7	14.5	16.8	57	8.2	8.9	9.9
23	12.6	14.3	16.5	58	8.1	8.7	9.6
24	12.4	14.1	16.3	59	8.0	8.6	9.4
25	12.3	14.0	16.1	60	7.9	8.4	9.2
26	12.1	13.8	15.9	61	7.7	8.2	8.9
27	12.0	13.6	15.6	62	7.6	8.1	8.7
28	11.8	13.4	15.4	63	7.4	7.9	8.5
29	11.7	13.2	15.2	64	7.3	7.7	8.3
30	11.6	13.1	15.0	65	7.1	7.5	8.0
31	11.4	12.9	14.8	66	6.9	7.3	7.8
32	11.3	12.7	14.6	67	6.7	7.1	7.6
33	11.2	12.6	14.4	68	6.6	6.9	7.4
34	11.0	12.4	14.2	69	6.4	7.7	7.1
35	10.9	12.3	14.1	70	6.2	6.5	6.9
36	10.8	12.1	13.9	71	6.0	6.3	6.7
37	10.6	11.9	13.7	72	5.8	6.1	6.5
38	10.5	11.8	13.5	73	5.6	5.9	6.2
39	10.4	11.6	13.3	74	5.4	5.6	5.9
40	10.3	11.5	13.2	75	5.2	5.4	5.6

* Simpson's Select Exercises, p. 260.

TABLE

TABLE II. *shewing the value of an annuity of* 1 *l. upon one* LIFE, *supposing the decrements of life to be equal.* *

Age.	Years purchase, at 5 *per cent.*	Ditto, at 4 *per cent.*	Ditto, at 3 *per cent.*	Age.	Years purchase, at 5 *per cent.*	Ditto, at 4 *per cent.*	Ditto, at 3 *per cent.*
8	14.54	16.79	19.74	43	11.43	12.68	14.19
9	14.61	16.89	19.87	44	11.29	12.50	13.96
10	14.61	16.89	19.87	45	11.14	12.32	13.73
11	14.54	16.79	19.74	46	10.99	12.14	13.49
12	14.48	16.69	19.60	47	10.84	11.94	13.25
13	14.41	16.61	19.47	48	10.68	11.75	13.01
14	14.34	16.51	19.33	49	10.52	11.55	12.76
15	14.27	16.41	19.19	50	10.35	11.34	12.51
16	14.19	16.31	19.05	51	10.18	11.14	12.26
17	14.12	16.21	18.91	52	9.99	10.92	11.99
18	14.05	16.11	18.76	53	9.82	10.70	11.73
19	13.97	15.99	18.61	54	9.63	10.48	11.46
20	13.89	15.89	18.46	55	9.44	10.25	11.18
21	13.81	15.79	18.31	56	9.24	10.02	10.90
22	13.73	15.67	18.15	57	9.04	9.77	10.62
23	13.64	15.55	17.99	58	8.83	9.53	10.33
24	13.56	15.44	17.83	59	8.61	9.28	10.03
25	13.47	15.32	17.66	60	8.39	9.02	9.73
26	13.38	15.19	17.49	61	8.16	8.75	9.42
27	13.28	15.07	17.33	62	7.93	8.49	9.12
28	13.19	14.95	17.15	63	7.69	8.21	8.79
29	13.09	14.82	16.98	64	7.44	7.92	8.46
30	12.99	14.68	16.80	65	7.18	7.63	8.13
31	12.89	14.55	16.62	66	6.92	7.33	7.79
32	12.70	14.41	16.44	67	6.43	7.03	7.45
33	12.77	14.27	16.25	68	6.36	6.71	7.09
34	12.57	14.13	16.06	69	6.07	6.39	6.74
35	12.45	13.98	15.86	70	5.78	6.07	6.38
36	12.33	13.83	15.67	71	5.47	5.73	6.01
37	12.21	13.68	15.47	72	5.15	5.38	5.63
38	12.09	13.52	15.26	73	4.83	5.03	5.25
39	11.97	13.36	15.05	74	4.49	4.67	4.85
40	11.84	13.19	14.84	75	4.14	4.29	4.45
41	11.71	13.03	14.63	76	3.78	3.91	4.01
42	11.58	12.86	14.41	78	3.42	3.52	3.63

* Dodson's Repository, vol. 2. p. 169.

Note. By increasing the numbers in table 11, by $\frac{2}{9}$ of a year's purchase,

purchase, to 54 years of age, by $\frac{3}{9}$ from 54 to 70, and by $\frac{4}{9}$ from 70 years and upwards, we shall obtain the present values *nearly* of an annuity (SECURED BY LAND) of 1 l. *per annum*, on a single life; supposing the decrements of life to be equal *.

148. In the case of a simple annuity, (147, Tables I. and II.) the heirs of the annuitant are not entitled to any thing, if the annuitant deceases before the term of payment; but in the case of an annuity secured by land, the heirs of the annuitant are entitled to receive a sum of money proportional to the time elapsed between the last payment of the annuity, and the time of the death of the annuitant: Thus, the value of a life of 40 years of age, at 4 l. *per cent.* compound interest, is 11.5, by Table I. and 13.19 years purchase, by Table II. and the value of an annuity secured by land for the same age and rate, is 13.412 years purchase nearly: Thus also, the value of a life of 66 years of age, at 4 l. *per cent.* compound interest, is 7.3 by Table I. and 7.33 by Table II and the value of an annuity secured by land, for the same age and rate, is 7.66 years purchase *nearly*.

PROPOSITION XXX.

149. If we take out *any* two of the homologous terms of an analogy, these two quantities, when considered by themselves, are said to constitute a GENERAL PROPORTION; one of these quantities being always *as* the other: And any one of the terms of a general proportion may be multiplied or divided by any number taken at pleasure.

Suppose the ratio $Q : q = R : r$, then to express the general proportion between Q and R, or between q and r, we put the quantities down in this manner, *viz.* $Q \because R$ and $q \because r$; and, by this notation, we mean nothing more than that the quantity Q is always *as* the homologous quantity R, and the quantity q *as* the homologous quantity r: Here then the quantities R, r may be either different values of the analogous quantities Q, q or considering Q, q as proper quantities, the quantities R, r may be improper quantities, supposing them referred to the proper quantities Q, q as their measures. Supposing (as above) the ratio $Q : q = R : r$, it is evident, that the quantity R will be greater, equal or less than the analogous quantity r, according as the quantity Q is greater, equal or less than the analogous quantity q: and therefore, considering the quantities Q, R as variable, they will either both together equally increase, or both together equally decrease. Thus, in every analogy, the homologous terms have a mutual dependence upon one another; and hence the propriety of the expression introduced on this sub-

* See Dodson's Repository, Vol. III.

ject by the illustrious * Sir Isaac Newton, that any term of an analogy is always AS its proper homologous term. Since then, by by supposition, the ratio $Q : q = R : r$, and since the ratio $R : r = \frac{m}{n} R : \frac{m}{n} r$ where m, n may represent any numbers whatever; therefore, according to the notation explained in this proposition, when $Q \cdot\cdot R$ it is also $Q \cdot\cdot \frac{m}{n} R$ (Ax. IV.)

150. Corol. 1. Suppose the ratio $Q : q = R : r = S : s$, &c. then according to the notation explained in this proposition we have $Q \cdot\cdot R$, $R \cdot\cdot S$, &c. as also $Q \cdot\cdot S$: Therefore, if one quantity Q is as another quantity R, and the quantity R as another quantity S, &c. then will the first quantity Q be as the last quantity S.

151. Corol. 2. Suppose the ratio $Q : q = R : r = S : s$; then, since the ratio $\overline{Q : q} = \overline{R \pm S : r \pm s}$ (13) as also ($2 \times \overline{Q : q} = \overline{R : r} + \overline{S : s}$ or) the ratio $Q^2 : q^2 = RS : rs$ (5) therefore, when $Q \cdot\cdot R$ and $R \cdot\cdot S$, it is $Q \cdot\cdot R \pm S$, as also $Q^2 \cdot\cdot RS$ or $Q \cdot\cdot \sqrt{RS}$. Thus, if one quantity Q be as a second quantity R, and the second quantity R as a third quantity S; then will the first quantity Q be as the sum or difference of the second and third quantities R, S, and also as the mean proportional between them (16.)

152. Corol. 3. Suppose the ratio $Q : q = R : r$; then since $\frac{m}{n}\overline{Q : q} = \frac{m}{n}\overline{R : r}$ (IX. Ax.) or the ratio $Q^{\frac{m}{n}} : q^{\frac{m}{n}} = R^{\frac{m}{n}} : r^{\frac{m}{n}}$ (5, 7): Therefore, when $Q \cdot\cdot R$ it is also $Q^{\frac{m}{n}} \cdot\cdot R^{\frac{m}{n}}$.—That is, both the terms of any general proportion may be anyhow equally involved or evolved.

153. Corol. 4. Suppose the ratio $\overline{Q : q} = \overline{R : r}$, and let $\overline{S : s}$ be any other ratio; then, since $\overline{Q : q} \pm \overline{S : s} = \overline{R : r} \pm \overline{S : s}$ (VII, VIII, Ax.) or the ratio $QS : qs = RS : rs$ (3), as also the ratio $\frac{Q}{S} : \frac{q}{s} = \frac{R}{S} : \frac{r}{s}$ (4); therefore when $Q \cdot\cdot R$ it is also $QS \cdot\cdot RS$ and $\frac{Q}{S} \cdot\cdot \frac{R}{S}$, supposing S to be any variable quantity whatever. Thus, both the terms of any general proportion may be multiplied or divided by any variable quantity taken at pleasure.

* Principia, B. I. Lem. 10. Scholium.

154. Corol. 5. Let Q, R, S, T, &c. be any number of variable quantities, and suppose that $Q \cdot\cdot RS$ and that $S \cdot\cdot T$, then will $Q \cdot\cdot RT$: For, since, by supposition $S \cdot\cdot T$, therefore $RS \cdot\cdot RT$ (153) and of consequence $Q \cdot\cdot RT$ (150). Thus, in any general proportion, instead of any variable quantity may be substituted any other *homologous* variable quantity.

155. Corol. 6. Let Q, R be two variable *homologous* quantities. Then, since by supposition, $Q \cdot\cdot R$, therefore, dividing both terms by R, we have $\frac{Q}{R} \cdot\cdot 1$ (153); but unity is a given invariable quantity, and therefore $\frac{Q}{R}$ is also a given invariable quantity, or $\frac{Q}{R}$ is always the same, let the quantities Q, R be any whatever variable quantities. Hence, in any general proportion, instead of any *invariable* quantity, we may always substitute unity.

156. Corol. 7. Suppose the ratio $Q:q = R:r$, and the ratio $S:s = T:t$; then, since $\overline{Q:q} \pm \overline{S:s} = \overline{R:r} \pm \overline{T:t}$ (VII, VIII Ax.) or the ratio $QS:qs = RT:rt$ (3) as also the ratio $\frac{Q}{S}:\frac{q}{s} = \frac{R}{T}:\frac{r}{t}$ (4): Therefore, when $Q \cdot\cdot R$ and $S \cdot\cdot T$ it is also $QS \cdot\cdot RT$ and $\frac{Q}{S} \cdot\cdot \frac{R}{T}$. That is, the terms of any general proportion may be multiplied or divided by the terms of any other general proportion.

157. Corol. 8. Suppose the ratio $Q:q = \frac{1}{V}:\frac{1}{v}$; then considering the homologous quantities, Q, $\frac{1}{V}$ as variable, they will either both together equally increase, or both together equally decrease: But, since V, $\frac{1}{V}$ are the reciprocals of one another, if the former increases the latter will diminish, and *vice versa*. Therefore, in considering two variable quantities, to determine their general proportion, if one of the quantities increases while the other equally diminishes, the former will always be *reciprocally* as the latter.

158. Corol. 9. Suppose the ratio $Q:q = \overline{R:r} + \overline{S:s}$; then if the ratio $S:s$ be a ratio of equality, the ratio $Q:q = R:r$, and if the ratio $R:r$ be a ratio of equality, the ratio $Q:q = S:s$. Therefore, if $Q \cdot\cdot R$ when S is invariable, and

$Q \cdot\cdot S$

$Q \propto S$ when R is invariable; then, when both the quantities R, S are variable, it is $Q \propto RS$.

159. Corol. 10. And in the same manner it may be demonstrated, that if $Q \propto R$ when V is invariable, and $Q \propto \frac{1}{V}$ when R is invariable; then when both the quantities V, R are variable, it is $Q \propto R \times \frac{1}{V}$.

160. Corol. 11. Let there be any cause A which produces the effect E in the time T; then, since we can judge of a cause by the effect which it can produce in a given portion of time, it is evident, that every effect produced in time depends not only upon its immediate efficient cause, but also on the time wherein the cause acts or exerts itself; and hence, by increasing or diminishing the time, the effect will be equally increased or diminished, allowing the cause to remain the same; and by increasing or diminishing the cause, the effect will be equally increased or diminished, supposing the time to remain the same. Thus, then $E \propto T$ when A is invariable, and $E \propto A$ when T is invariable; therefore universally, or when both A, T are variable, it is $E \propto AT$. That is, the effect produced is always in the compound ratio of the efficient cause and the time.

161. Corol. 12. Since then universally $E \propto AT$, therefore $E \times \frac{1}{A} \propto T$ and $E \times \frac{1}{T} \propto A$. That is, the time of producing an effect is as the said effect directly, and its efficient cause reciprocally; and any efficient cause is directly as the effect which it can produce, and reciprocally as the time.

162. Corol. 13. If we suppose the effect (E) to be invariable, then $AT \propto 1$ (155) and of consequence $A \propto \frac{1}{T}$ or $T \propto \frac{1}{A}$.— Thus, with regard to a *given* effect, the efficient cause is reciprocally as the time, or the time reciprocally as the efficient cause.

163. Corol. 14. In the general theorem (160) $E \propto AT$, suppose that $A \propto T$; then in this case, $E \propto A^2$ or $E \propto T^2$. That is, when the efficient cause is as the time, the effect produced will be in the duplicate ratio of the efficient cause, or in the duplicate ratio of the time.

FINIS.

NOTES.

A.—PROPOSITION I.

164. ADDITION being the collecting together of quantities into one sum or aggregate (Def. IV.), preserving their signs, and uniting such as can be united (Prop. I.); it is evident, that quantities expressed by different symbols can be represented only AS added together, by being brought into one expression under their signs: And like quantities being multiples of some same quantity (Def. XII.), these, when affected with the same or like signs, are added together *by adding the coefficients, to the sum adjoining the common quantity, and prefixing the common sign*; and when affected with unlike signs, *by subtracting the less coefficient from the greater, to the remainder adjoining the common quantity, and prefixing the sign of the greater.*

165. As every quantity contains itself *once* (Def. VI, VII. Corol. I.), any integral quantity, therefore, may be expressed as a fraction, by giving to it unity as a denominator; and any fraction may be expressed as the numerator of a fraction, having unity for its denominator also.

166. And as a fraction is nothing more than an expression of a quotient (Def. VII.), the value of a fraction, therefore, is not affected by any equal multiplication or division of both its terms. And hence the various *reductions* of fractions from this and the former simple principles (165).

167. FRACTIONS of one and the same denomination being like parts of their respective wholes or numerators (Def. VII.), are added together *by adding the numerators, and underwriting the common denominator* (Ax. III.). And all fractions may be reduced to equivalent fractions of one and the same denomination, by *taking*, according to notation (Def. VI.), *the product of all the denominators for the common denominator, and the product of each numerator into all the denominators but its own for the new numerators*, the terms of the fractions being equally multiplied by this operation (166).

168. *Ratios* are added together as in Proposition (3.), and according to Definition XVIII. Corollaries I, II. Thus $\overline{A:B}+\overline{C:D}=\overline{\frac{A}{B}:1}+\overline{\frac{C}{D}:1}\times=\frac{A}{B}\times\frac{C}{D}:1=\frac{A}{B}\times\frac{C}{D}=A\times C:B\times D$ *.

B.—PROPOSITION II.

169. SUBTRACTION determining the difference of quantities (Def. V.), is therefore the converse of addition (Def. IV.), and takes place absolutely by *changing the sign or signs of the ablative quantity*, finally abbreviating the general expression of the *difference* by uniting such terms as can be united (Prop. I.).

* The quantities $\frac{ab}{b}=a$, and $\frac{cd}{d}=c$ (Def. VII. Corol. I, II.), may represent any two fractions; and the product ac (Def. VI.) is evidently equivalent to $\frac{acbd}{bd}$ (Def. VII.), the product under the numerators and denominators.

170. FRACTIONS of one and the same denomination are subtracted one from another, by subtracting the *one numerator from the other, and underwriting the common denominator.*

171. RATIOS are subtracted one from another, as in Proposition I. and according to Definition XVIII. Corollaries I, II. Thus:

Since by Prop. I. $\overline{A:B}+\overline{C:D}=AC:BD$; therefore, taking away the ratio $C:D$ from each of these equals, and the ratio $A:B=\overline{AC:BD}-\overline{C:D}$ (Ax. IV.), or the ratio $\frac{A}{B}:1=\frac{AC}{BD}:1-\frac{C}{D}:1$; and these ratios having now the common consequent, unity, we have only to observe what operation must obtain among the antecedents $\frac{AC}{BD}, \frac{C}{D}$ to bring out the antecedent $\frac{A}{B}$: And here it is manifest, that the product of the numerator by D, and of the denominator by C of the antecedent $\frac{AC}{BD}$, gives directly the antecedent $\left(\frac{A\times CD}{B\times CD}=\right)\frac{A}{B}$ (Def. VI. and Corol. I.); but $\frac{D}{C}$ is the inverse or reciprocal of the antecedent $\frac{C}{D}$ (Def. XI. Corol. I.) Therefore, in taking away or subtracting one ratio from another ratio, we must *change the sign of the ablative ratio, and invert the terms thereof*, according to the Proposition (4).

C.—PROPOSITION III.

172. MULTIPLICATION being a manifold addition, repeating one quantity so often as the same or another quantity contains units (Def. VI.). Hence (1.) a simple quantity may be multiplied by a simple quantity, *by taking the product of the quantities or literal symbols, according to notation* (Def. VI.), *and thereto prefixing the product of the coefficients for the coefficient of the product.* (2.) A compound quantity may be multiplied by a simple quantity, *by beginning at the left hand, multiplying the simple quantity into each term of the multiplicand successively*, as in Case I. *and making the products* affirmative *or* negative, *according as the factors have* like *or* unlike *signs.* And (3.) one compound quantity may be multiplied by another compound quantity, by placing the multiplier under the multiplicand, term under term, that being considered as the multiplicand which hath the greater number of terms; then, *beginning at the left hand, each term of the multiplier is drawn or multiplied into each term of the multiplicand successively*, as in Case II. the different products being placed under one another, with like quantities under like quantities when there are such quantities: And the *whole* product is obtained by collecting together all the different products into one sum (Prop. I.).

173. FRACTIONS are multiplied together, by taking the *product of the numerators for the numerator of the* product, *and the product of the denominators for the denominator of the* product; as demonstrated, Note A (168). Or the same may be demonstrated in manner following, viz. Suppose $\frac{A}{B}=P$, and $\frac{C}{D}=Q$ (Def. VII. Corol. II.), or $A=BP$, and $C=DQ$

(Definition

(Definition VII. Corol. I.); then $AC = BP \times DQ$ (Ax. IX.) $= PQ \times BD$ (Def. VI.), and consequently $\frac{AC}{BD} = \frac{PQ \times BD}{BD}$ (Ax. X.) $= PQ$ (Def. VII.) $= P \times Q = \frac{A}{B} \times \frac{C}{D}$ by supposition.

174. HENCE, in multiplying together fractions, the denominators of any of the factors may be interchanged to depress the final product. And hence also, a fraction may be multiplied by an integral quantity, by multiplying the numerator *by the integral quantity*, or by dividing the *deuominator thereby*.

D.—PROPOSITION IV.

175. DIVISION being a manifold subtraction, and the converse therefore of multiplication, determines how often one quantity may be taken out of, or is contained in, another quantity, by considering the dividend as a product; whereof the efficient factors are the divisor and quotient (Def. VII.). And hence (1.) a simple quantity may be divided by a simple quantity, by inquiring *what quantity, multiplied into the divisor, produces the dividend*, or *by placing the dividend above a small line, and the divisor under it, and* expunging all *numbers and quantities that are common efficients of each*. (2.) A compound quantity may be divided by a simple quantity, by beginning at the left hand, dividing as in Case I. each term of the dividend by the divisor, and *making the quotients affirmative or negative, according as the signs of the divisor and dividend are* like *or* unlike. And (3.) one compound quantity may be divided by another compound quantity, by ranging the terms of the divisor and dividend, according to the dimensions of any the same quantity contained in each, and proceeding as in common arithmetic; the quotient quantities being determined by *dividing the first term of the* dividend, *and of every* divdiual, *by the first term of the divisor*.

176. FRACTIONS are divided one by another, by multiplying the *inverse* or *reciprocal* (Def. XI. Corol. I.) of the fractional divisor into the fractional dividend, as demonstrated, Note B (171). Or the same may be demonstrated in manner following, viz. Suppose $\frac{A}{B} = P$, and $\frac{C}{D} = Q$ (Def. VII. Corol. II.), or $A = BP$, and $C = DQ$ (Def. VII. Corol. I.), and therefore $AD = BPD$, and $BC = BDQ$ (Ax. IX.); then $\frac{AD}{BC} = \frac{BPD}{BDQ}$ (Ax. X.) $= \frac{P}{Q}$ (Def. VII.) $= P \div Q = \frac{A}{B} \div \frac{C}{D}$ by supposition.

177. And hence, a fraction may be divided by an integral quantity, by dividing the *numerator by the integral quantity*, or by multiplying the *denominator thereby*.

E.—PROPOSITION VII.

178. SINCE (Def. XVIII. Corol. I.) the ratio $A : mA = B : mB$ $(= 1 : m)$, and the ratio $mA : A = mB : B$ $(= m : 1)$; where A, mA, B, mB, or mA, A, mB, B, may represent *any* four analogous magnitudes; and where m may represent *any* number integral, fractional, &c. Therefore, in

 any

any analogy, the first term has as much magnitude when compared with the second, as the third when compared with the fourth; and, consequently, if the first term be greater, equal, or less than the second, the third term will be *equally* greater, equal, or *equally* less than the fourth.

179. And hence all the *theorems* in this Proposition (13), and Corollaries thereof (14, 15, 16,) as in the following table, viz.

$$\underline{A : mA = B : mB \; (= 1 : m)}$$

$$\left.\begin{array}{l} \qquad\quad A \times mB = mA \times B \\ A + mA : A = B + mB : B \; (= 1 + m : 1) \\ A + mA : mA = B + mB : mB \; (= 1 + m : 1) \end{array}\right\} \text{Jointly.}$$

$$\left.\begin{array}{l} A \backsim mA : A = B \backsim mB : B \; (= 1 \backsim m : 1) \\ A \backsim mA : mA = B \backsim mB : mB \; (= 1 \backsim m : m) \end{array}\right\} \text{Disjointly.}$$

$$\left.\begin{array}{l} A \substack{+\\ \backsim} mA : A = B \substack{+\\ \backsim} mB : B \; (= 1 \substack{+\\ \backsim} m : 1) \\ A \substack{+\\ \backsim} mA : mA = B \substack{+\\ \backsim} mB : mB \; (= 1 \substack{+\\ \backsim} m : m) \end{array}\right\} \text{Conversion.}$$

$$A + mA : A \backsim mA = B + mB : B \backsim mB \; (= 1 + m : 1 \backsim m) \text{ Mixtim.}$$

$A : B = mA : mB$, Alternation or Permutation.

$mA : A = mB : B \; (= m : 1)$ Inversion.

$$\left.\begin{array}{l} A : mA \\ B : mB \end{array}\right\} = A + B : mA + mB \; (= 1 : m) \text{ Syllepsis.}$$

$$\left.\begin{array}{l} A : mA \\ B : mB \end{array}\right\} A \backsim B : mA \backsim mB \; (= 1 : m) \text{ Dialepsis.}$$

Hyp. $A : mA = B : mB = C : mC = D : mD$, &c.

Then $A : mA = A + B + C + D, \&c. : mA + mB + mC + mD, \&c. \; (= 1 : m)$.

180. Suppose now universally, the ratio $A : B = C : D$; then, since $\frac{B}{A} \times A = B$ (Def. VIII. Corol. I.), therefore $\frac{B}{A} \times C = D$ (178) the fourth proportional to A, B, C. And hence the common arithmetical rule for finding a fourth proportional, by dividing the product of the second and third by the first analogous term.

181. And because $\frac{B}{A} \times C = D$, when the ratio $A : B = C : D$ (180); therefore, by equal multiplication ($B \times C = A \times D$ or) $AD = BC$ (Ax. IX.) as before (178). Or this may be demonstrated in manner following, viz. Since $A : B = C : D$ (Hyp.), therefore $\overline{A : B} - \overline{C : D} = 0$ (Ax. II.); that is, $\overline{A : B} + \overline{D : C}$, or $AD : BC = 0$ (Prop. II.), and consequently $AD = BC$ (Def. XVIII. Corol. III.). Thus, from an analogy, may be derived an equation between the product of the extremes and means; and conversely, as shown in the third Corollary of this Proposition (16).

182. As the terms of a ratio must necessarily be of one and the same kind or denomination, and as there may be the same ratio of magnitude between two quantities or numbers of one kind or denomination, as between

two quantities or numbers of any other kind of denomination (Def. XVIII.): Hence the very general and extensive application of the doctrine of PROPORTION in all arts, sciences, business, &c. And in every case of proportion, to determine truly a fourth proportional to three given, the proportionals must be so ranged, that the two *given* quantities, or numbers of one and the same kind, be the terms of the first ratio, and the third given quantity, or number of the same kind, with that required; the antecedent of the second ratio, the consequent being the fourth or required term, which may always be represented by some one of the final letters of the alphabet, x, y, z, v, &c.

183. Then, as things of *greater* value require a *less* number, and things of *less* value a *greater* number, to make an equal exchange; and as the less the number of things the *greater* must be their value, and the *greater* the number of things the *less* their value, to make them equal. Therefore, consider from the nature of the case proposed, whether the third term be greater, equal, or less than the fourth or required term, x, y, &c. and range the terms of the first ratio in the same order of magnitude respectively; or make the first term greater, equal, or less than the second term, according as the third term is greater, equal, or less than the fourth or required term, x, y, &c. And if any terms are mixed, or consist of different denominations, they may be reduced to the least assigned denomination when necessary, and the analogy restated.

184. The analogy being stated as directed (183); if either the first and second terms, or the first and third terms * be divisible by any the same quantities or numbers, instead of these terms take the respective quotients, and restate as often as may be the analogy in terms of the quotients: Then finding the multiplier of the first term producing the second (180), and the same will be the multiplier of the third term producing the fourth or required term, x, y, &c. (178).

F.—PROPOSITION XV. SCHOLIUM I.

185. In the equation $px-x^2=rq$, when $x=\frac{1}{2}p$, then $\frac{1}{4}p^2=rq$; and in every other case or value of x, $\frac{1}{4}p^2$ is *greater* than rq. For, suppose $x=\frac{1}{2}p\pm s$, where $s > s\ \frac{1}{2}p$; then by substitution ($px-x^2=$) $\frac{1}{4}p^2-s^2=rq$, and of consequence $\frac{1}{4}p^2 > r\ rq$. But though the root of the equation $px-x^2=rq$ in general terms admits of two different interpretations or values, as demonstrated in this Scholium, and though as in the examples to this case of the Proposition both these values may be true, examples, however, may occur in which one only of the values may be taken; and the value to be taken will be determined from the structure of the equation, or from the operation producing the equation. Suppose $\sqrt{\frac{1}{2}x}=21-x$, where x is evidently less than 21; then (Ax. XI.) $\frac{1}{2}x$ ($=\overline{21-x}^2$) $=441-42x+x^2$ (Prop. V.), and hence $42\frac{1}{2}x-x^2=441$; where $x=18$, the only true value.

* Suppose the ratio A : B = C : D; then, by adding the ratio m : 1 to each, the ratio mA : B = mC : D (Ax. VII.); where m may represent any number, integral, fractional, &c.

$$42\tfrac{1}{2}x - x^2 = 441$$

Here $\frac{42\frac{1}{2}}{2} = 21\tfrac{1}{4}$, and $\overline{21\tfrac{1}{4}}^{\,2} = 451\tfrac{9}{16}$

$$\overline{21\tfrac{1}{4}}^{\,2} - \overline{42\tfrac{1}{2}x - x^2}\left(= 451\tfrac{9}{16} - 441\right) = 10\tfrac{9}{16}$$

$$\overline{21\tfrac{1}{4}}^{\,2} - 42\tfrac{1}{2}x + x^2\left(= 10\tfrac{9}{16}\right) = \tfrac{169}{16}$$

$$21\tfrac{1}{4} - x\left(= \tfrac{13}{4}\right) = 3\tfrac{1}{4}$$

$$x\left(= 21\tfrac{1}{4} - 3\tfrac{1}{4}\right) = 18.$$

Or

$$\overline{21\tfrac{1}{4}}^{\,2} - 42\tfrac{1}{2}x + x^2\left(= 10\tfrac{9}{16}\right) = 9 + 1\tfrac{1}{2} + \tfrac{1}{16}$$

$$21\tfrac{1}{4} - x = 3\tfrac{1}{4}$$

$$x = 18.$$

G—PROP. XV. SCHOL. III. or ART. XXX.

186. The theorem here given for evolving a quadratic binomial, may be demonstrated *synthetically*, in manner following, viz.

Since $\overline{B \pm \sqrt{B^2 - Q}}^{\,2} = B^2 \pm 2B\sqrt{B^2 - Q} + B^2 - Q$ (Proposition V.) $= 2B^2 - Q \pm \sqrt{4B^2 \times \overline{B^2 - Q}}$ (Prop. IV.), and since (Hyp.) $\frac{R+Q}{2} = B^2$, and of consequence $\left(R - \frac{R+Q}{2} =\right) \frac{R-Q}{2} = B^2 - Q$, as also $\sqrt{R^2 - S} = Q$; therefore, by restitution, $\overline{B \pm \sqrt{B^2 - Q}}^{\,2} = R \pm \sqrt{R^2 - Q^2} = R \pm \sqrt{R^2 - \overline{R^2 - S}} = R \pm \sqrt{S}$ (Prop. XI.), according to the Theorem (30). Thus: the square of the greater part of the root is $\frac{R+Q}{2}$, or $R + \sqrt{R^2 - S}$; and that of the less part of the root $\frac{R-Q}{2}$, or $R - \sqrt{R^2 - S}$.

187. And hence, universally, the root of a quadratic binomial may be determined from the sum and difference of the roots of half the sum and half the difference of the leading terms of the binomial, and the root of the difference of the squares of the terms.

Ex. 3. Where $R = 3$, and $S = 8$. Here $Q = 1$, $B = \sqrt{2}$, and $B \pm \sqrt{B^2 - Q} = \sqrt{2} \pm 1$.

Ex. 4. Where $R = 6$, and $S = 20$. Here $Q = 1$, $B = \sqrt{5}$, and $B \pm \sqrt{B^2 - Q} = \sqrt{5} \pm 1$.

And to Example 1. may be added the five following, making *six binomials in the tenth book of Euclid.*

Binomial I. $4 + \sqrt{11}$, root of $27\sqrt{704}$.

Binomial II. $\sqrt[4]{12} + \sqrt[4]{\tfrac{27}{4}} - 6 + \tfrac{1}{2}\sqrt{147}$, *former bimedial.*

Binomial III. $\sqrt[4]{\tfrac{80}{3}} + \sqrt[4]{15} - \tfrac{245}{3} + \sqrt{80}$, *latter bimedial.*

Binomial IV. $\sqrt{\tfrac{7}{2} + \tfrac{1}{2}\sqrt{29}} + \sqrt{\tfrac{7}{2} - \tfrac{1}{2}\sqrt{29}} - 7 + \sqrt{20}$, the *major.*

Binomial V. $\sqrt{\sqrt{5} + 1} + \sqrt{\sqrt{5} - 1} - 7 + \sqrt{40}$, the *potent, a rational with a medial.*

Binomial VI. $\sqrt{\sqrt{5} + \sqrt{3}} + \sqrt{\sqrt{5} - \sqrt{3}} - \sqrt{20} + \sqrt{8}$, the *potent two medials.*

H.

H.—PROPOSITION XVI.

188. Since in a right angled plane triangle the tangent of half either angle at the hypothenuse is to the radius, as the side opposite to the angle to the hypothenuse and other side taken together: And *inversely* (14), radius, to the tangent of half either angle at the hypothenuse, as the side opposite to the angle to the excess of the hypothenuse above the other side (5. e. 2). Hence, to the analytical and geometrical resolution of quadratic equations in the last and in this Proposition, may be added the *trigonometrical* resolution of the same, as in the following practical rules, viz.

189. Rule I. Increase half the logarithm of the absolute term (rq) by the number 10, and from the sum subtract the logarithm of half the coefficient (p) of the second term; the remainder in Cases I. and II. is the *logarithmic tangent*, and in Case III. the *logarithmic sine* of an angle A.

Rule II. To and from half the logarithm of the absolute term (rq), add and subtract the *logarithmic tangent* of half the angle A; let the sum be diminished and the difference increased by the number 10, and the numbers resulting will be the logarithms of the *roots* of the equations in Cases I. and II.

Rule III. From and to half the logarithm of the absolute term (rq), subtract and add the logarithmic tangent of half the angle A; let the difference be increased and the sum diminished by the number 10, and the numbers resulting will be the logarithms of the *two roots* of the equation in Case III.

Case. I. Example $x^2+347x=22110$, to determine x.

Here $p=347$, and $rq=22110$.

The logarithm of rq	22110=	4.3445887
From half thereof	+10	=12.1722943
Take the logarithm of $\frac{1}{2}p$	$\frac{347}{2}$=	2.2392995
Remains logarithmic tangent of angle A .	40° 35′ 52″=	9.9329948
To logarithmic tangent of $\frac{1}{2}$ angle A . .	20° 17′ 56″=	9.5680714
Add $\frac{1}{2}$ the logarithm of rq	22110=	2.1722943
The sum (—10) is the logarithm of x	=	1.7403657

And $\because x=55$.

Case II. Example $x^2-\frac{110}{24}x=\frac{1300}{24}$, to determine x.

Here $p=\frac{110}{24}$, and $rq=\frac{1300}{24}$.

Preparation.

From log. of 1300=5.1139434	From log. of (half 110=) 55=1.7403627
Take log. of 24=1.3802112	Take log. of . . . 24=1.3802112
Rem. log. rq $\frac{1300}{24}$=1.7337322	Remains log. of $\frac{1}{2}p$. $\frac{55}{24}$=0.3601515

From

From half the logarithm of $rq+10$ . . . $\frac{1300}{24}+10=10.8668661$
Take the logarithm of $\frac{1}{2}p$ . . . . $\frac{55}{24}=0.3601515$

Remains log. tangent of angle A . . . . $72^\circ\ 42'\ 16''=10.5067146$

From half the log. of $rq+10$ . . . . . . $\frac{1300}{24}+10=10.8668661$
Take log. tangent of $\frac{1}{2}$ angle A . . . . $36^\circ\ 21'\ 8''=9.8668617$

Remains the logarithm of . . . . $x=1.0000044$
And $\therefore x=10$.

Case III. Example $\frac{264}{25}x-x^2=\frac{695}{25}$, to determine the values of x.
Here $p=\frac{264}{25}$, and $rq=\frac{695}{25}$.

PREPARATION.

From log of $695=2.8419848$ From log. of . $(\frac{264}{2})131=2.1205739$
Take log. of $25=1.3979400$ Take log. of . . . $25=1.3979400$

Rem. log. $rq\ \frac{695}{25}=1.4440448$ Rem. log. of $\frac{1}{2}p$ . . $\frac{132}{25}=0.7226339$

From half the log. of $rq+10$ . . . . . . $\frac{695}{25}+10=10.7220224$
Take log. of $\frac{1}{2}p$ . . . . $\frac{132}{25}=0.7226339$

Remains log. sine of angle A . . . . . $86^\circ\ 57'\ 45''=9.9993885$

To and from half the log. rq . . . . $\frac{695}{25}=0.7220224$
Add and subtract the log. tang. of $\frac{1}{2}$ angle A $43^\circ\ 28'\ 52''=9.9769646$

The log. of $x=\begin{cases}\text{Sum}-10 \ldots\ldots = 0.6989870\\ \text{Diff.}+10 \ldots\ldots = 0.7450578\end{cases}$

And $\therefore x=5$, and 5.56*.

I.—PROPOSITION XVII. AND COROLLARIES.

190. This Proposition (32) may be demonstrated *synthetically*, in manner following, viz. Suppose $y=\sqrt[3]{r+\sqrt{r^2q^3}}$, and $v=\sqrt[3]{r-\sqrt{r^2-q^3}}$; then $z=y+v$ (Hyp.), and $z^3=\overline{y+v}^3$ (Ax. XI.) $=y^3+3y^2v+3yv^2+v^3$ (Prop. V.) $=y^3+3yv\times\overline{y+v}+v^3=y^3+v^3+3yv\times z$ (Hyp.); but, since $y=\sqrt[3]{r+\sqrt{r^2-q^3}}$, and $v=\sqrt[3]{r-\sqrt{^2-q^3}}$ (Hyp.), therefore $yv=\sqrt[3]{r+\sqrt{r^2-q^3}\times r-\sqrt{r^2-q^3}}$ (Prop. III.) $=\sqrt[3]{r^2-r^2-q^3}=\sqrt[3]{q^3}=q$ (Prop. VI.), and $y^3+v^3=r+\sqrt{r^2-q3}+r-\sqrt{r^2-q3}=2r$ (Pr. I.): Whence, by restitution z^3 $(=y^3+v^3+3yv\times z)=2r+3qz$, and consequently $z^3-3qz-2r=0$, according to the Proposition.

In the equation of Corollary II. viz. $z^3+3qz-2r=0$ (34), where $z>s\sqrt[3]{2r}$, and also $z>s\frac{2r}{3q}$; and where $r-\sqrt{r^2+q^3}$ is negative, because $\sqrt{r^2+q^3}>r\ r$, as is obvious: Therefore, in this equation

* See the first of DR HALLEY's four Lectures on the Geometrical construction of equations, published as an Appendix to KERSEY's Algebra; and also DODSON's Repository, vol. i. pp. 115, 119, 123.

$$z=\sqrt[3]{r+\sqrt{r^2+q^3}}-\sqrt[3]{-r+\sqrt{r^2+q^3}}$$

Or the value of z is as in the other expression of this Corollary, by one evolution of the cubic root.

192. In the equation of Corollary III. viz. $z^3-3q\,z+2r=0$ (35), or $3qz-z^3=2r$; since $3qr-z^3=2r$, it is evident that $z > s\sqrt{3q}$: Suppose therefore $z=\sqrt{q}$; then by substitution, $2q\sqrt{q}=2r$. Now, as z is always less than $\sqrt{3q}$, and may be equal to $\sqrt{q}$; so it may fall short of $\sqrt{q}$ by a quantity (s) less than $\sqrt{q}$, and it may exceed $\sqrt{q}$ by a quantity (s), which when added to $\sqrt{q}$, shall give a sum less than $\sqrt{3q}$; and in both these cases, the quantity $2r$ is less than $2q\sqrt{q}$ (or r less than $q\sqrt{q}$): For, by supposing $z=\sqrt{q}\pm s$, we have by substitution $2q\sqrt{q}-3\,\overline{\sqrt{q}\pm s}\times s^2=2r$, where it is evident that $2q\sqrt{q} > r\ 2r$, or $q\sqrt{q} > r\ r$; and therefore, by equal involution, $q^3 > r\,r^2$.

193. And of the general complete equation of Corollary IV. (36), the third and fourth terms may evidently be represented as both affirmative or both negative, or as one affirmative and the other negative.

$$x^3-3px^2-\overline{3q-3p^2}\times x+\overline{3pq-p^3-2r}=0.$$

K.—PROPOSITION XX. AND SCHOLIUM. (49, 50).

194. This Prop. (49) for evolving a cubic binomial, may be demonstrated in manner following. Since $\overline{B+\sqrt{B^2-Q}}^3=\overline{4B^3-3BQ}+\overline{4B^2-Q}\times\sqrt{B^2-Q}$ (Prop. V.) $=4B^3-3BQ\pm\sqrt{\overline{4B^2-Q}^2\times\overline{B^2-Q}}=4B^3-3BQ\pm\sqrt{16B^6-24B^4Q+9B^2Q^2-Q^3}$ (Proposition III.) $=4B^3-3BQ\pm\sqrt{\overline{4B^3-3BQ}^2-Q^3}$ (Prop. VI.), and since by supposition ($8B^3-6BQ=2R$ or) $4B^3-3BQ=R$, as also $\sqrt[3]{R^2-S}=Q$ or $R^2-S=Q^3$; therefore, by restitution, $\overline{B\pm\sqrt{B^2-Q}}^3$ $(=R\pm\sqrt{R^2-\overline{R^2-S}})=R\pm\sqrt{S}$.

195. And that $z^2+2mz+4m^2-3q=0$, as in the Scholium (50), will appear directly on performing the division according to Proposition IV.; it being considered, that as $z=2m$ (Proposition XVII.), so $8m^3-6qm-2r=0$.

$$
\begin{array}{l}
z-2m)\,z^3-3qz-2r\,(z^2+2mz+4m^2-3q\\
\quad z^3-2mz^2\\
\hline
\qquad 2mz^2-3qz\\
\qquad 2mz^2-4m^2z\\
\hline
\qquad\quad 4m^2z-3qz\\
\qquad\quad 4m^2z-8m^3\\
\hline
\qquad\qquad -3qz+8m^3\\
\qquad\qquad -3qz+6qm\\
\hline
\qquad\qquad\qquad 8m^3-6qm-2r\ (=0).
\end{array}
$$

L.—PROPOSITION XXIII. (67).

196. Since by supposition $A+Bx+Cx^2+Dx^3+$&c.$=a+bx+cx^2+dx^3+$&c. therefore, by transposition (19) $\overline{A-a}+\overline{B-b}\times x+\overline{C-c}\times x^2+\overline{D-d}\times x^3+$&c.$=0$; if now ($A-a=0$ or) $A=a$, then $\overline{B-b}\times x+\overline{C-c}\times x^2+\overline{D-d}\times x^3+$&c.$=0$ *universally*, or whatever may be the value of x. But the quantity $\overline{B-b}\times x+\overline{C-c}\times x^2+\overline{D-d}\times x^3+$&c. evidently vanishes not only when x itself vanishes, but also when the several coefficients $B-b$, $C-c$, $D-d$, &c. vanish: And thus, the series $Bx+Cx^2+Dx^3+$ &c., $bx+cx^2+dx^3+$&c. being mutually convertible, or the same, must therefore coalesce into one when $B-b=0$, $C-c=0$, $D-d=0$, &c. and this single series will always converge when x is a converging quantity. And though, in the general equation of this Proposition, all the terms are represented as affirmative, they may be any how affected with the signs of addition and subtraction.

M.—PROPOSITION XXIV. (70).

197. Since m, n are affirmative integers by supposition; therefore, by division (Prop. IV. and Corol.), $\frac{v^m-z^m}{v-z}=v^{m-1}+v^{m-2}z+v^{m-3}z^2$ (m), or continued to m terms (A), and in the same way $\frac{x^n-y^n}{x-y}=x^{n-1}+x^{n-2}y+x^{n-3}y^2$ (n); whence, supposing ($x^n=v^m$, $y^n=z^m$ or) $x=v^{\frac{m}{n}}$, $y=z^{\frac{m}{n}}$; then, since x^{n-1} ($=v^{\frac{m}{n}\times\overline{n-1}}$) $=v^{m-\frac{m}{n}}$, x^{n-2} ($=v^{\frac{m}{n}\times\overline{n-2}}$) $=v^{m-\frac{2m}{n}}$, &c. (Proposition V.); therefore, by substitution, $\left(\frac{x^n-y^n}{x-y}=\right)\frac{v^m-z^m}{v^{\frac{m}{n}}-z^{\frac{m}{n}}}=$(B) $v^{m-\frac{m}{n}}+v^{m-\frac{2m}{n}}z^{\frac{m}{n}}+v^{m-\frac{3m}{n}}z^{\frac{2m}{n}}(n)$

But $\left(\frac{v^m-z^m}{v-z}\div\frac{v^m-z^m}{v^{\frac{m}{n}}-z^{\frac{m}{n}}}=\right)\frac{v^{\frac{m}{n}}-z^{\frac{m}{n}}}{v-z}\left(=\frac{A}{B}\right)=$

$$\frac{v^{m-1}+v^{m-2}z+v^{m-3}z^2\ (m)}{v^{m-\frac{m}{n}}+v^{m-\frac{2m}{n}}z^{\frac{m}{n}}+v^{m-\frac{3m}{n}}z^{\frac{2m}{n}}(n)}$$

and dividing both sides of this equation by $v^{\frac{m}{n}-1}$ (Ax. X.) we have $\frac{v^{\frac{m}{n}}-z^{\frac{m}{n}}}{v-z}\times v^{\overline{\frac{m}{n}-1}}=$

$$\frac{v^{m-1}+v^{m-2}z+v^{m-3}z^2\ (m)}{v^{m-1}+v^{m-\frac{m}{n}-}z^{\frac{m}{n}}+v^{m-\frac{2m}{n}-1}z^{\frac{2m}{n}}\ (n)}=$$

$$\frac{1+\frac{z}{v}+\overline{\frac{z}{v}}\Big|^2+\overline{\frac{z}{v}}\Big|^3\ (m)}{1+\overline{\frac{z}{v}}\Big|^{\frac{m}{n}}+\overline{\frac{z}{v}}\Big|^{\frac{2m}{n}}+\overline{\frac{z}{v}}\Big|^{\frac{3m}{n}}\ (n)}$$ by dividing both numerator aud deno-minator by v^{m-1} (Prop. IV.); and hence $\frac{v^{\frac{m}{n}}-z^{\frac{m}{n}}}{v-z}=v^{\frac{m}{n}-1}\times$

$$\frac{1+\frac{z}{v}+\overline{\frac{z}{v}}\Big|^2+\overline{\frac{z}{v}}\Big|^3\ (m)}{1+\overline{\frac{z}{v}}\Big|^{\frac{m}{n}}+\overline{\frac{z}{v}}\Big|^{\frac{2m}{n}}+\overline{\frac{z}{v}}\Big|^{\frac{3m}{n}}}$$

N.—PROPOSITION XXV. (80).

198. In the *pure* equation $y^m=N$, suppose as in the Proposition $\frac{A}{B}$ to be a near approximation to the value of y; and the general *formula* is

$$\frac{N+\overline{m-1}\times\overline{\frac{A}{B}}\Big|^m}{m\times\overline{\frac{A}{B}}\Big|^{m-1}} \quad \text{Or} \quad \frac{NB^m+\overline{m-1}\times A^m}{mA^{m-1}B}$$

O.—PROPOSITION XXVI. AND COROL. IV. (86, 90).

199. These series, for the logarithms of $\frac{1}{1-n}$, and $1+n$, may be derived in the most simple and obvious manner from the following observations, viz. (1.) In every system of logarithmic numbers, the logarithm of *unity* is nothing. (2.) The sums and differences of any number of logarithmic quantities, are themselves logarithmic quantities. (3.) The product P of any quantity A by a logarithmic quantity L, is always a logarithmic quantity; for since $A\times L=P$ (Hyp.), the ratio $1:A=L:P$ (Prop. VII. Cor. III.); and L being a logarithmic quantity by supposition, it is evident that the *analogous* quantity P is of the same kind (Def. IV. e. 5). And hence, (4.) The quantity $A\times L$ being always a logarithmic quantity, it may therefore be assumed as *a* logarithmic of A; wherefore, resolving the quantity A by division (Prop. IV.), into its constituent parts, $a\pm b\pm c\pm d\pm e\pm$, &c. and supposing K, M, N, P, Q, &c. to be logarithmic quantities, giving Ka, Mb, Nc, Pd, Qe, &c. logarithms of a, b, c, d, e, &c. respectively, we shall have this general expression in a series for defining the logarithm of any quantity A, viz.

$Ka + Mb + Nc + Pd + Qe +$ &c.

200. And hence, Case I. To determine the series for the logarithm of $\frac{1}{1-n}$, or for the measure of the *ratio* $\frac{1}{1-n} : 1$.

Here $A \left(= \frac{1}{1-n}\right) = 1 + n + n^2 + n^3 + n^4 +$ &c. (Prop. IV.) $= a + b + c + d + e +$ &c. by supposition; whence, since $a = 1$, therefore $K = 0$ (Obs. 1.), and of consequence $L, \frac{1}{1-n} = Mn + Nn^2 + Pn^3 + Qn^4 +$ &c. or $2L, \frac{1}{1-n} = 2Mn + 2Nn^2 + 2Pn^3 + 2Qn^4 +$ &c. But by the principles of logarithms, $2L, \frac{1}{1-n} = L, \left.\overline{\frac{1}{1-n}}\right|^2 = L, \frac{1}{1-2n-n^2}$ (Prop. V. Cor. II.), therefore, in the series for the $L, \frac{1}{1-n}$, if the quantity $2n-n^2$ be substituted instead of n, we shall have $L, \left.\overline{\frac{1}{1-n}}\right|^2 (= M \times \overline{2n-n^2} + N \times \overline{2n-n^2}^2 + P \times \overline{2n-n^2}^3 + Q \times \times \overline{2n-n^2}^4 +$ &c.) $= 2M \times n + \overline{4N - M} \times n^2 + \overline{8P - 4N} \times n^3 + \overline{16Q - 12P + N} \times n^4 +$ &c. by expanding the terms $= 2M \times n + 2N \times n^2 + 2P \times n^3 + 2Q \times n^4 +$ &c. as above, and of consequence (Prop. XXIII.) $N = \frac{1}{2}M$, $P = \frac{1}{3}M$, $Q = \frac{1}{4}M$, &c. (67): Thus, $2L, \frac{1}{1-n} = 2Mn + Mn^2 + \frac{2}{3}Mn^3 + \frac{2}{4}Mn^4 +$ &c. and therefore $L \frac{1}{1-n} = M \times \overline{n + \frac{1}{2}n^2 + \frac{1}{3}n^3 + \frac{1}{4}n^4 +}$ &c.

201. Case II. To determine the series for the logarithm of $1 + n$, or for the measure of the *ratio* $1 + n : 1$.

Since the ratio proposed is $1 + n : 1$, its reciprocal or inverse will be $1 : 1 + n$ (Prop. VII. Corol. I.) or $\frac{1}{1+n} : 1$; therefore, since $\frac{1}{1+n} = 1 - n + n^2 - n^3 + n^4$ &c. (Proposition IV.) proceeding as in Case I. we shall have $\left(L, \left.\overline{\frac{1}{1+n}}\right|^2 = 2L, \frac{1}{1+n} = \right) - 2M \times n + \overline{4N - M} \times n^2 - \overline{8P - 4N} \times n^3 + \overline{16Q - 12P + N} \times 4^4$, &c. $= -2M \times n + 2N \times n^2 - 2P \times n^3 + 2Q \times n^4$, &c. and hence $L, \frac{1}{1+n} = -M \times \overline{n - \frac{1}{2}n^2 + \frac{1}{3}n^3 - \frac{1}{4}4^4}$, &c. or $L, 1 + n = M \times \overline{n - \frac{1}{2}n^3 + \frac{1}{3}n^3 - \frac{1}{4}n^4}$, &c. (Prop. II.)

202. And as by the series in Corollary IV. (90), the number $1 \mp n$ may be determined from its logarithm L being given, and that whatever may be the relation of magnitude or quantity between L the logarithm and M the *modulus* of the system; if, therefore, it be considered that equality, sameness, or identity, is the most simple relation that may obtain between the quantities L, M (Ax. I.), upon this supposition the ratio $1 + n : 1 = 2 + \frac{1}{2} + \frac{1}{2.3} + \frac{1}{2.3.4} +$, &c. $: 1 = 2.7182818459$, &c. $: 1$; or the ratio $1 : 1 - n = 1 : \frac{1}{2} - \frac{1}{2.3} + \frac{1}{2.3.4} -$ &c. $= 1 : 0.367879441171$, &c.:

&c.: And either of these numeral ratios is that called by the learned MR COTES, the *ratio modularis*; the *modulus* or module being always the measure of this ratio in every logarithmic system.

P.—PROP. XXVIII. COROL. II. ART. 122, 124, 125, and 126.

203. The general simple interest equation $\overline{tr+1}\times p=s$ (122), where r is the interest of L 1. for 1 year, called the *ratio of the rate*, as being always the 100th part of the rate *per cent. per annum*, applies in every instance of any sum of money paid *after* or *before* due. In the first case p is the principal and s the amount thereof, and its interest trp for time t and ratio of rate r. In the second case, supposing p, s to change places, the equation becomes $\overline{tr+1}\times s=p$: And here, s is the *present worth* or *value* of the principal p for time t and ratio of rate r; or s is the principal which in time t and at ratio of rate r would amount to the sum or principal p. And as the excess of the amount above the principal is the *interest*; so the excess of the principal above the present worth is the *rebate* or *discount*.

204. If, therefore, the interest of any given principal for any given time and rate, expressed in pounds and decimal places when there are places, be divided by the yearly amount of L 1. for the same rate, the quotient will be the rebate of said principal for the given time and rate. And if the *yearly* interest of any principal, at any rate *per cent.*, be divided by the yearly amount of L 1. at the same rate, the quotient will be the yearly rebate of said principal.

205. In the *third* and *fourth* lines of the table (124), the values of r, t are wrong printed, and should have been expressed in manner following, viz.

s, t, a	$r=\dfrac{\frac{s}{t}-a}{\frac{t-1}{2}a-s}$
s, r, a	$t=\sqrt{\dfrac{2s}{ar}\ \overline{\dfrac{1}{r}-\dfrac{s}{a}+\dfrac{1}{2}}^{\,2}}-\overline{\dfrac{1}{r}-\dfrac{s}{a}+\dfrac{1}{2}}$

206. The method of computing *accurately* the interest of any sum or principal for any number of days not greater than (a year or) 365, as given in article 125, being *general* for any whatever rate of interest, and not confined or limited, as there stated, to the rate of 5 per cent.; the general calculation, therefore, of interest for any time or number of days not greater than a year, and at any whatever rate *per cent. per annum*, may be performed with great facility in manner following, viz.

207. *Let the product under the sum or principal, the number of days*, and the double of the rate per cent. *be advanced five places to the right; multiply this number by 3, advancing the product one place to the right: Subtract the latter advanced product from the former, advancing the remainder one place to the right; and the sum of the two advanced products and advanced remainder, diminished by itself when advanced four places to the right, will be the true interest in every case.*

The

208. The example for illustrating the general simple interest table (126) being quite perverted by the *Printer*, is here given properly corrected as follows, viz. Taking the former example (125).

$$4239,767 \times 5 \times 256 = 5426,901.76$$

Then	50000	L 136	19	8	2.85
	4000	10	19	2	0.55
	200	0	10	11	2.30
	60	0	3	3	1.81
	9	0	0	5	3.67
	Interest required,	L 148	13	7	3.18

209. And in using this general table (126) to the greatest advantage, supposing two additional columns on the right hand as below; *let the product of principal rate and time,* when divided by 100, *be resolved into its component parts, and the corresponding interest, numbers, taken from the table, and added together, will give in the sum total, the interest required.*

Two additional columns to the general table (126).

No.	Q.	No.	Q.
1	2.63	.05	.13
.9	2.37	.04	11
.8	2.10	.03	.08
.7	1.84	.02	.05
.6	1.58	.01	.026
.5	1.32	.009	.004
.4	1.05	008.	.002
.3	.75	.007	.018
.2	.53	.006	.015
.1	.26	.005	.013
.09	.24	.004	.012
.08	.21	.003	.009
.07	.18	.002	.005
.06	.16	.001	.003

210. But the following, perhaps, is the best general and perfectly accurate method of computing the interest of money for any number of days not greater than 365, at any whatever rate *per cent. per annum*, and equally easy with the common method at the rate of 5 *per cent.* only, viz.

Divide the product of the sum or principal, the number of days, and the double of the rate, by 73000, *and the quotient will be the interest accurately in every case.*

211. And hence, to determine the rebate or discount at any rate of interest, of any sum or principal paid any number of days not greater than 365 before due; *divide the product of the sum or principal, the number of days,* and the double of the rate, *by* 73000 *increased by the product of the number of days* and double of the rate, *and the quotient will be the* rebate *accurately in every case.*

Q.—PROP. XXIX. COROL. XIII. XV. ART. 140, and 147.

212. The EUCLIDEAN series determining all PERFECT NUMBERS, as given in Corollary XIII. (140) being quite perverted by the *Printer*, is here reprinted accurately as follows, viz.

$s = 1$	$1 + 2.\overline{2-1}$	$1 + 2.\overline{2^2-1}$	$1 + 2.\overline{2^4-1}$	$1 + 2.\overline{2^6-1}$	$1 + 2.\overline{2^8-1}$	$1 + 2.\overline{2^{10}-1}$
$2^{n-1} = 1$	2^{1}	2^2	2^4	2^6	2^8	2^{10} &c.
$n = 1$	2	3	5	7	9	11 &c.

213. And the latter part of article 147, which should here come in after the colon (:) in the 6th line, having been omitted by the *Printer*, is here added as containing an useful Theorem, in the following terms, viz.

In the following tables, if N represent the tabular number corresponding to any age, the value of the annuity, &c. a will be $a \times N$ (146); and subtracting this from the perpetuity $\frac{a}{r}$ (145), the value of the *reversion* of a fee-simple estate, &c. (a), after a single life of a given age will be had, viz. $a \times \overline{\frac{1}{r} - N}$.

FINIS.

ERRATA.

Article 168, line 1, read in *the* Proposition.

———— —— 1, Note (*) read $\frac{cd}{d}=c$

—— 171, —— 1, read *Proposition II.*

—— 181, —— last, read *shown.*

—— 187, —— 2, read of *the* roots.

—— 189, —— 1, read to *determine* the values of x.

SYNOPSIS

OF

BOOK V.

OF

EUCLID'S ELEMENTS,

ACCORDING TO

DR SIMSON'S EDITION.

ST ANDREWS:

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M.DCC.XCVII.

PREFACE.

THIS SYNOPSIS, first drawn up according to Keil's edition of Commandine's Euclid, was afterwards changed, where neceſſary, according to Dr Simſon's edition of the Elements, and is now publiſhed with the addition of a very ſimple demonſtration of Propoſition VIII. from the learned Mr Playfair's Elements of Geometry, for the uſe of Students, who too generally neglect the truly elegant demonſtrations of Euclid, in this moſt uſeful part of the Elements, and thereby obſtruct their improvement in the other branches of Mathematics; and in Natural Philoſophy, very conſiderably.

DEFINITIONS.

DEFINITION I.

MATHEMATICAL ANALYSIS is the ſcience of QUANTITY in general; where by QUANTITY is underſtood whatever is meaſurable, or made up of parts.

DEFINITION II.

QUANTITY is divided into PROPER and IMPROPER. That is called *proper quantity* which can be meaſured by its own *kind*, and that *improper quantity* which cannot be meaſured by its own *kind*, but to which a meaſure may be aſſigned by means of ſome *proper quantity* having connection with it *.

DEFINITION III.

In MATHEMATICAL ANALYSIS all quantities are repreſented by alphabetical letters. And the ALGORITHM or ARITHMETIC of quantity admits of the ſame operations as that of *numbers*, viz. addition, ſubtraction, multiplication, diviſion, involution, and evolution.

DEFINITION IV.

ADDITION collects or gathers together quantities into one *ſum* or *aggregate*. And the operation of addition is *denoted* by interpoſing between the quantities to be added together the *ſign* of addition (+ *plus*).

DEFINITION V.

SUBTRACTION determines the *difference* of quantities. The quantity from which the ſubtraction is made is called the *minuend*, and that by which ſubtraction is made the *ſubtrahend* or *ablative quantity*. And the operation of ſubtraction is *denoted* by interpoſing between the minuend and ſubtrahend the *ſign* of ſubtraction (— *minus*).

DEFINITION VI.

MULTIPLICATION compounds or repeats one quantity as often as there are UNITS in the ſame, or any other quantity. The quantity ariſing by multiplication

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* Vid. Philoſophical Tranſactions, No. 489, for year 1748.

tiplication is called a *product*, the quantities multiplied together being the *factors* of the product; whereof one is termed the *multiplier*, and the other the *multiplicand*. And the operation of multiplication is *denoted* by interposing between the factors the *sign* of multiplication (× *into*); or simply by the *juxtaposition* of the factors.

DEFINITION VII.

Division decompounds or resolves a product into its two factors or efficient parts, one of them being always assigned. The product to be resolved by division is called the *dividend;* the factor by which division is made, the *divisor;* and that arising by division, the quotient. And the operation of division is *denoted* by interposing between the dividend and divisor the *sign* of division (÷ *by*); or by placing the dividend *above* a straight line, and the divisor *under* it; the upper quantity or dividend being then termed the *numerator*, and the lower quantity or divisor the *denominator* of the fraction or quotient.

COROLLARY I.

A quotient, then, in reference to its divisor and dividend, is universally the quantity, which, multiplied into the divisor, produces the dividend (Def. VI.).

COROLLARY II.

A fractional expression, in reference to its denominator, or the product of the denominator by any integral quantity, is the numerator of the fraction, or the product of the numerator by the same integral quantity *.

DEFINITION VIII.

Involution compounds or repeats one quantity *continually*, as often as there are units in itself, and in all subsequent products; and these products are called powers, the generating quantity being the radix of the scale or *first* power. And the operation of involution is *denoted* by placing, at the top of the root upon the right hand, a small numeral figure, expressive of the number of repetitions of the radix at any place or term of the series; which numeral figure is emphatically called the index, exponent, or characteristic of the power.

DEFINITION

* Thus, in reference to 4, 8, 12, 16, 20, 24, 28, &c. $\frac{1}{4}$ is 1, 2, 3, 4, 5, 6, 7, &c. and $\frac{3}{4}$, therefore, 3, 6, 9, 12, 15, 18, 21, &c. and the same with regard to any other fraction.

DEFINITION

DEFINITION IX.

EVOLUTION decompounds or resolves POWERS into ROOTS, which being the converse of involution (Def. VIII.), is therefore effected by *continual* division. And the operation of evolution is *denoted* in the Newtonian method by giving to the quantity as an *index unit, divided by the exponent of the particular power* to be resolved: Or the same may be denoted in the common way by placing over the quantity the RADICAL SIGN having the INDEX of the *particular power* prefixed thereto ($\sqrt[m]{}$).

COROLLARY.

Since multiplication (Def. VI.) and division (Def. VII.) are only repeated addition (Def. IV.) and repeated subtraction (Def. V.); and since involution (Def. VIII.) and evolution (Def. IX.) are only repeated multiplication and repeated division; therefore ADDITION and SUBTRACTION are the two PRIMARY or FUNDAMENTAL operations in the algorithm of quantity (Def. III.); and of consequence the SIGNS of addition and subtraction the two *capital signs* in the analysis of quantity (Def. I.).

DEFINITION X.

SIMPLE QUANTITIES are those expressed by single symbols. COMPOUND QUANTITIES are those expressed by different symbols, connected together by the signs of addition and subtraction: And in every Expression of compound quantities, the connecting *sign* belongs to the quantity before which it is placed; any quantity being considered as *affirmative* or *negative* according as it is affected with the sign of addition or subtraction: A quantity without any sign being always affirmative.

DEFINITION XI.

A NUMERAL FIGURE, whether integral or fractional, prefixed to any quantity, is called a COEFFICIENT or FELLOW-FACTOR, as *denoting* how often the quantity is reckoned; and the quantity itself, with its coefficient prefixed, is called a MULTIPLE. A quantity with a fractional coefficient, is sometimes called SUBMULTIPLE. And when no number is prefixed to any quantity, UNITY is understood as the COEFFICIENT.

COROLLARY I.

Hence, assuming any quantity A; since $A = 1 \times A$ (Def. VI.) and $\frac{1}{A} \times A = 1$ (Def. VII. Corol. I.); that is, since A is the same multiple of unity as $\frac{1}{A}$ is part or submultiple of unity, the quantities A, $\frac{1}{A}$, are there-

fore

ſore ſaid to be INVERSE or RECIPROCAL of one another. And hence, in the Newtonian notation of evolution (Def. IX.), the index or exponent of the root is the inverſe or reciprocal of the index or exponent of the power to be evolved.

COROLLARY II.

The coefficient of a fraction, if a whole number, may be prefixed to the numerator of the fraction, and if a fractional number, the numerator of the coefficient may be prefixed to the numerator of the fraction, and the denominator of the coefficient to the denominator of the fraction. For a fraction being an expreſſion of a quotient (Def. VII.) to repeat a fraction according to the units in any number (Def. VI.), is to multiply the number into the numerator, the denominator remaining the ſame: And to take any part of a fraction according to the units in any number, or to divide a fraction by a number, is to multiply the number into the denominator, the numerator remaining the ſame.

DEFINITION XII.

SIMILAR or LIKE QUANTITIES are thoſe expreſſed by the ſame ſymbol or ſymbols equally repeated; and DISSIMILAR or UNLIKE QUANTITIES are thoſe expreſſed by different ſymbols, or by the ſame ſymbol or ſymbols differently repeated.

DEFINITION XIII.

A STRAIGHT LINE drawn above any compound quantity, and conſolidating as it were the compound quantity into one maſs, is called a VINCULUM. And by means of the VINCULUM, the ſums, differences, products, quotients, powers, and roots of compound quantities, are *denoted* in the ſame manner as thoſe of ſimple quantities.

DEFINITION XIV.

The EQUALITY of quantities perfectly equivalent to one another, is *denoted* by interpoſing between the equivalent quantities the ſign of equality (=): and every expreſſion of equivalent quantities denoted in this manner, is called an EQUATION. And when there are different quantities connected together by the *ſigns* of addition and ſubtraction, on both ſides of an equation, theſe are called TERMS or MEMBERS of the equation.

DEFINITION XV.

The PRINCIPAL QUANTITY in an equation is that quantity whereof the VALUE or VALUES depend upon, or flow from the *values* of all the other

quantities

quantities combined any how in the terms of the equation. This quantity is denominated the ROOT of the equation; the VARIABLE, the INDETERMINATE, or the UNKNOWN quantity; and is always *denoted* or represented by some one of the final letters of the alphabet.

DEFINITION XVI.

THOSE QUANTITIES in an equation whereof the VALUES depend not upon the combination and arrangement of the terms, but continue always the *same*, let them be combined any how, are called INVARIABLE, or DETERMINATE, or KNOWN, or GIVEN quantities; and these are always *denoted* or represented by some of the *initial letters* of the alphabet.

DEFINITION XVII.

REDUCTION OF AN EQUATION brings the principal quantity (Def. XV.) to possess one side of the equation *alone*, and totally disengaged from other quantities; and the PRINCIPAL QUANTITY is then DETERMINED, as being expounded on the other side of the equation by a quantity equivalent thereto, in consequence of that combination of the INVARIABLE QUANTITIES (Def. XVI.) from which the equation derived its *origin*.

COROLLARY.

Any quantity in an equation may be considered as the PRINCIPAL QUANTITY, and may be enunciated by REDUCTION; a value possible and consistent in all respects being *previously* substituted instead of the *principal quantity* wherever it appears in the equation.

DEFINITION XVIII.

RATIO is a mutual relation of two magnitudes of the same kind to one another in respect of quantity: Or according to Euclid in the Elements *, *RATIO is a certain mutual habitude of two magnitudes of the same kind, according to quantity:* That is, the mutual comparison of two homogeneous quantities according to magnitude or quantity, is called RATIO, and is *denoted* by interposing between the quantities two points placed vertically (:). The quantities compared together are called the terms of the ratio; the former quantity being denominated the *antecedent*, and the latter the *consequent:* And the ratio is denominated a ratio of *majority*, or of greater inequality, *equality*, or of equal to equal, and *minority* or of less inequality, according as the antecedent is greater, equal, or less than the consequent.

COROLLARY

* Book V. Definition III.

COROLLARY I.

Since any quantity is equal to *unity* repeated as often as there are units in said quantity (Def. VI.), it therefore follows that any quantity may be considered, and hence *denoted* as a *ratio* having unity for a consequent: And because any quantity may be considered as a *unity* of its own kind, it follows also that a ratio undergoes no change of VALUE, by taking instead of the terms of the ratio, any EQUIMULTIPLES, integral or fractional of the same (Def. XI.). Thus in the ratio m A : A, considering A as 1, then m A as m (Def. XI.), and the ratio m A : A the same with the ratio m : 1; thus also, in the ratio m A : n A, considering A as 1, then m A as m, n A as n, and the ratio m A : n A the same with the ratio m : n. And hence all ratios may be reduced to *equivalent* ratios having unity for a common consequent, by taking the general expressions of the quotients of the antecedents by the consequents (Def. VII.) as the antecedent.

COROLLARY II.

EQUAL RATIOS obtain between adjacent terms, and UNEQUAL RATIOS between terms that are *not* adjacent in any *regular* series of powers (Def. VIII.) or roots (Def. IX.); and also in any series of quantities where all terms of the series are EQUIMULTIPLES of immediately *contiguous* terms; reckoning from either extremity of the series, and taking any quantity, integral or fractional, as the multiplier *. In the series of integral terms increasing, or fractional terms decreasing, the product of two assumed terms, in reference to the higher assumed term, is always a subsequent or higher term, and the quotient of the higher assumed term, divided by the lower, a preceding or lower term. Whence, since any term may be considered as a ratio, whereof the consequent is unity (Corol. I.), it is evident that these ratios are *generated* from a multiplication or division of their antecedents; the former operation generating *greater*, and the latter *less* ratios or terms; therefore, considering every ratio as a quantity *sui generis*, the former operation in the arithmetic of ratios is called ADDITION, and the latter SUBTRACTION.

COROLLARY

* 1, 2, 4, 8, 16, 32, 64, 128, 156, &c.
1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, $\frac{1}{32}$, $\frac{1}{64}$, $\frac{1}{128}$, $\frac{1}{256}$, &c.
1, 3, 9, 27, 81, 144, 739, 2187, 6591, &c.
1, $\frac{1}{3}$, $\frac{1}{9}$, $\frac{1}{27}$, $\frac{1}{81}$, $\frac{1}{243}$, $\frac{1}{724}$, $\frac{1}{2187}$, $\frac{1}{6591}$, &c.

&c. &c.

1, m A, $m\,m$ A, $m\,m\,m$ A, $m\,m\,m\,m$ A, $m\,m\,m\,m\,m$ A, &c.

&c. &c.

COROLLARY III.

Between *equal* quantities there can be *no* ratio, accurately ſpeaking. The idea of equality being that of ſameneſs or identity, and ſo oppoſed to the idea of compariſon, as implying that of diverſity. Hence a ratio of equality is ſaid to be *nothing*, or to have *no* meaſure; and converſely, of a ratio equal to *nothing*, the terms are equal. Hence,

COROLLARY IV.

Aſſuming any quantity and itſelf again, or which is the ſame, aſſuming two equal quantities, and ſtating the ratio in terms, let the *conſequent remain invariably the ſame*. Then if the antecedent be ſuppoſed to increaſe continually, or to flow upward through all degrees of magnitude, the ſeries will comprehend *all* ratios of majority or greater inequality, and theſe *increaſing* continually as the ſeries expands continually: but if the antecedent be ſuppoſed to diminiſh continually, or to flow downwards through all inferior degrees of magnitude, the ſeries will comprehend *all* ratios of minority or leſs inequality, and theſe *increaſing* continually as the ſeries expands downwards continually. Thus the ratio of equality, though properly ſpeaking *no* ratio, is the *common limit* between ratios of majority and ratios of minority: and the *magnitude* of any ratio may be eſtimated by its *diſtance* from the common limit or ratio of equality, a ratio being *greater* the *farther* it is removed from the ratio of equality: And from the *different ways* by which *all* ratios of majority and minority are derived, in the ſeries expanded upwards and downwards indefinitely, it appears that theſe ratios are quite *oppoſite in kind*, or heterogeneous to one another, and ſo incapable of any mutual compariſon as to magnitude or quantity.

COROLLARY V.

Hence, in comparing together ratios as to equality and inequality, if equimultiples of both terms of the ratios, compared together by the alternate conſequents or antecedents, give the ſame ratio or equal ratios, the ratios compared together are equal (Corol. I.); and otherwiſe unequal: And of unequal ratios, that univerſally is the greater ratio which is moſt remote or diſtant from the ratio of equality (Corol. IV.).

$$\begin{array}{l}\text{Ratios} \\ \begin{array}{r} 3:2 \\ 81:54 \end{array}\left\{\begin{array}{l} 3:2=3\times54:2\times54=162:108 \\ 81:54=81\times2:54\times2=162:108 \end{array}\right\}\end{array}\begin{array}{c}\text{Equal} \\ 3:2=81:54\end{array}$$

$$\begin{array}{l}\text{Ratios} \\ \begin{array}{r} 5:7 \\ 115:161 \end{array}\left\{\begin{array}{l} 5:7=5\times115:7\times115=575:805 \\ 115:161=115\times5:161\times5=575:805 \end{array}\right\}\end{array}\begin{array}{c}\text{Equal} \\ 5:7=115:161\end{array}$$

Ratios

Ratios		Unequal
7 : 5	7 : 5 = 7×9 : 5×9 = 63 : 45	7 : 5 >r 11 : 9
11 : 9	11 : 9 = 11×5 : 9×5 = 55 : 45	45 : 45....55 : 45....63 : 45

Ratios		Unequal
5 : 7	5 : 7 = 5×11 : 7×11 = 55 : 77	5 : 7 >r 9 : 11
9 : 11	9 : 11 = 9× 7 : 11× 7 = 63 : 77	77 : 77....63 : 77....55 : 77

SCHOLIUM.

This method of comparing together ratios of minority as to inequality, or as greater or less, differs from that given by EUCLID, VII Def. e. v. And from this definition of EUCLID, as applied to ratios of minority, it might be inferred, admitting, or supposing negative quantities, that —2 is greater than —18, for example, though in truth but the 9th part of the same. EUCLID, it must be observed, considers a ratio of minority as greater the nearer it is to, and not the farther it is from, the ratio of equality; it being obvious that 3, for example, has more magnitude when compared with 5, than it has when compared with 17; and that, therefore, the ratio 3 : 5 considered abstractly, is greater than the ratio 3 : 17. But, considering the ratios 3 : 5, 3 : 17, as what they really are, namely, as ratios of minority, the ratio 3 : 5 is evidently a less ratio of *minority* than the ratio 3 : 17, and accordingly has a less *logarithmic* measure.

DEFINITION XIX.

The INEQUALITY of quantities being either in excess or defect, is *denoted* accordingly, by interposing between the unequal quantities the proper sign of inequality (>r *greater* or >s *less*).

DEFINITIONS.

DEFINITION I.

A leſs magnitude is ſaid to be a part of a greater magnitude, when the leſs meaſures the greater; that is, when the leſs is contained a certain number of times exactly in the greater.

Thus A is a part of *m* A, as being contained *m* times exactly in *m* A, *m* being a whole number.

DEFINITION II.

A greater magnitude is ſaid to be a multiple of a leſs, when the greater is meaſured by the leſs; that is, when the greater contains the leſs a certain number of times exactly.

Thus *m* A is a multiple of A by *m*; becauſe *m* A contains A *m* times exactly, *m* being a whole number.

DEFINITION III.

Ratio is a certain mutual habitude, or compariſon of two magnitudes of the ſame kind, as to quantity or magnitude.

SCHOLIUM.

A ratio is expreſſed by interpoſing between the magnitudes compared together, or the terms of the ratio, two points placed vertically (:) *; the former term of the ratio being called the* antecedent, *and the latter the* conſequent. *Thus, the expreſſion A : B denotes the ratio of A to B, or the abſolute relation as to magnitude of the antecedent A to the conſequent B, or of the conſequent B to the antecedent A, and that whether this relation be aſſignable or not.*

When the antecedent of a ratio exceeds the conſequent, the ratio is ſaid to be a ratio of greater inequality *or of* majority ; *when the antecedent of a ratio is exceeded by the conſequent, the ratio is ſaid to be a ratio of* leſs inequality *or of* minority ; *and when the antecedent of a ratio neither exceeds nor is exceeded by the conſequent, the ratio is ſaid to be a ratio of* equal *to* equal *or of* equality.

DEFINITION IV.

Magnitudes are ſaid to have a ratio to one another, when the leſs can be multiplied ſo as to exceed the other.

COROLLARY I.

As of two magnitudes the less cannot, by any multiplication whatever, exceed the other, unless the magnitudes are of one and the same kind, or homogeneous; so, between heterogeneous magnitudes there can be no ratio.

COROLLARY II.

As of two magnitudes there can be no distinction of less or greater, unless the magnitudes be unequal; so, between equal magnitudes there can be no ratio properly so called: A ratio of equality, therefore, is said to be nothing, or equal to nothing; and conversely, of a ratio equal to nothing, the terms are equal.

DEFINITION V.

The first of four magnitudes is said to have the same ratio to the second which the third has to the fourth, when any equimultiples whatsoever of the first and third being taken, and any equimultiples whatsoever of the second and fourth. If the multiple of the first be less than that of the second, the multiple of the third is also less than that of the fourth; or if the multiple of the first be equal to that of the second, the multiple of the third is also equal to that of the fourth; or if the multiple of the first be greater than that of the second, the multiple of the third is also greater than that of the fourth.

Let A, B, C, D, be four magnitudes, and m, n, any whatever numbers.

$$A, B, C, D, \quad \left.\begin{matrix}A\\C\end{matrix}\right\} m = \left\{\begin{matrix}mA\\mC\end{matrix}\right. \quad \left.\begin{matrix}B\\D\end{matrix}\right\} n = \left\{\begin{matrix}nB\\nD\end{matrix}\right.$$

$$A, B, C, D, \left\} \begin{matrix}\text{If } mA > r = \text{or} > s\ nB\\ \text{and } mC > r = \text{or} > s\ nD\end{matrix} \right\{ \text{The ratio } A : B = C : D$$

Thus, of the four numbers 3, 2, 81, and 54, the ratio $3 : 2 = 81 : 54$.

$$\left.\begin{matrix}3\\81\end{matrix}\right\} 5 \quad \begin{matrix}15\\405\end{matrix} \Bigg| \quad \left.\begin{matrix}2\\54\end{matrix}\right\} 7 \quad \begin{matrix}14\\378\end{matrix}$$

Here $15 > r\ 14$, and $405 > r\ 378$

$$\left.\begin{matrix}3\\81\end{matrix}\right\} 6 \quad \begin{matrix}18\\486\end{matrix} \Bigg| \quad \left.\begin{matrix}2\\54\end{matrix}\right\} 9 \quad \begin{matrix}18\\486\end{matrix}$$

Here $3 \times 6 = 2 \times 9 (= 18)$, and $81 \times 6 = 54 \times (9 = 486)$

$$\left.\begin{matrix}3\\81\end{matrix}\right\} 8 \quad \begin{matrix}24\\648\end{matrix} \Bigg| \quad \left.\begin{matrix}2\\54\end{matrix}\right\} 14 \quad \begin{matrix}28\\756\end{matrix}$$

Here $24 > s\ 28$, and $648 > s\ 756$

And the same will appear according to any multiplication whatever. And thus also, of the four numbers 5, 7, 115, and 161, the ratio $5 : 6 = 115 : 161$.

$$\left.\begin{array}{r}5\\115\end{array}\right\}10 \quad \begin{array}{r}50\\1150\end{array}\Bigg| \quad \left.\begin{array}{r}7\\161\end{array}\right\}6 \quad \begin{array}{r}42\\966\end{array}$$

Here $50 > r\ 42$, and $1150 > r\ 966$.

$$\left.\begin{array}{r}5\\115\end{array}\right\}7 \quad \begin{array}{r}35\\805\end{array}\Bigg| \quad \left.\begin{array}{r}7\\161\end{array}\right\}5 \quad \begin{array}{r}35\\805\end{array}$$

Here $5 \times 7 = 7 \times 5 (= 35)$, and $15 \times 7 = 161 \times 5 (= 805)$

$$\left.\begin{array}{r}5\\115\end{array}\right\}10 \quad \begin{array}{r}50\\1150\end{array}\Bigg| \quad \left.\begin{array}{r}7\\161\end{array}\right\}12 \quad \begin{array}{r}84\\1932\end{array}$$

Here $50 > s\ 84$, and $1150 > s\ 1932$.

And the ſame will appear according to any multiplication whatever.

SCHOLIUM.

To compare together as to equality, in the eaſieſt manner, numeral ratios, or ratios expreſſed in numeral terms; let them be reduced to equivalent ratios in one and the ſame ſeries, by taking equimultiples of the terms of each ratio by the alternate conſequents or antecedents: And if the ratios thus reduced to, or brought to be in one and the ſame ſeries, be expreſſed by the ſame terms, the ratios compared together are equal to one another.

Ratios. Reduced to the ſame ſeries. Equal.

$$\begin{array}{r}3:2\\81:54\end{array}\left\{\begin{array}{l}3:2 = 3 \times 54 : 2 \times 54 = 162 : 108\\81:54 = 81 \times 2 : 54 \times 2 = 162 : 108\end{array}\right\} 3:2 = 81:54$$

Ratios. Reduced to the ſame ſeries. Equal.

$$\begin{array}{r}5:7\\115:161\end{array}\left\{\begin{array}{l}5:7 = 5 \times 115 : 7 \times 115 = 575 : 805\\115:161 = 115 \times 5 : 161 \times 5 = 575 : 805\end{array}\right\} 5:7 = 115:161$$

DEFINITIONS VI, VIII, XII.

Magnitudes which have the ſame ratio are called proportionals; and the terms of two ratios are ſaid to compoſe an *analogy*, which is uſually expreſſed by ſaying, the firſt magnitude is to the ſecond as the third is to the fourth. And in an analogy the two antecedents are called *homologous* to one another, as alſo the two conſequents; but the terms of the ſame ratio are called *analagous* to one another.

DEFINITION VII.

When of the equimultiples of four magnitudes (taken as in Def. V.) the multiple of the firſt is greater than that of the ſecond, but the multiple of the third is not greater than the multiple of the fourth; then the firſt is ſaid to have to the ſecond a greater ratio than the third magnitude has to the fourth; and, on the contrary, the third is ſaid to have to the fourth a leſs ratio than the firſt has to the ſecond.

A : B,

$$\begin{array}{llll} & A : B,\ C : D & & \textit{Then} \\ \left.\begin{matrix}A\\C\end{matrix}\right\}m & \left.\begin{matrix}B\\D\end{matrix}\right\{n & \left\{\begin{matrix} m\,A \gt r\ n\,B \\ m\,C\ not \gt r\ n\,D\end{matrix}\right\} & \begin{matrix}A : B \gt r\ C : D\\ C : D \gt s\ A : B\end{matrix}\end{array}$$

SCHOLIUM.

In this definition for comparing together unequal ratios, it is assumed by Euclid, *that a ratio of majority is greater the* farther *it is removed from, and a ratio of minority greater the* nearer *it is to, the ratio of equality : Therefore, numeral ratios, or ratios expressed in numeral terms, of the same kind, and belonging to the same series, may be compared together as to greater or less, by considering only their distances from the ratio of equality ; a ratio of majority being greater the* farther *it is removed from, and a ratio of minority greater the* nearer *it is* ~~removed~~ *to, the ratio of equality.*

$1 : 5,\ 2 : 5,\ 3 : 5,\ 4 : 5,\ \overline{5 : 5},\ 6 : 5,\ 7 : 5,\ 8 : 5,\ 9 : 5,$ &c.
&c. $5 : 9,\ 6 : 9,\ 7 : 9,\ 8 : 9,\ \underline{9 : 9},\ 10 : 9,\ 11 : 9,\ 12 : 9,\ 13 : 9,$ &c.

In these series, the ratios on the right hand of the limit or ratio of equality, are ratios of majority regularly increasing, *and those on the left hand ratios of minority regularly* decreasing, *according to* Euclid.

When ratios are not in the same series, they may be reduced to one and the same series, and compared together as above, by taking equimultiples of the terms of each ratio by the alternate consequents or antecedents (Def. V. Scholium).

$$\left.\begin{array}{l} \textit{Unequal ratios.} \\ 7 : 5,\ 11 : 9 \\ 7 : 5 = 7\times 9 : 5\times 9 = 63 : 45 \\ 11 : 9 = 11\times 5 : 9\times 5 = 55 : 45 \\ \underline{\textit{And} \because 7 : 5 \gt r\ 11 : 9} \\ \underline{45 : 45}\cdots\cdots 55 : 45 \cdots\cdots 63 : 45 \end{array}\right\} \quad \left\{\begin{array}{l} \textit{Unequal ratios.} \\ 5 : 7,\ 9 : 11 \\ 5 : 7 = 5\times 11 : 7\times 11 = 55 : 77 \\ 9 : 11 = 9\times 7 : 11\times 7 = 63 : 77 \\ \underline{\textit{And} \because 5 : 7 \gt s\ 9 : 11} \\ \underline{77 : 77}\cdots\cdots 63 : 77 \cdots\cdots 55 : 77 \end{array}\right.$$

And in comparing together unequal ratios, according to this definition of Euclid, *the multiples to be taken are the terms of the less ratio* inverted, *or any numbers continually approximating to, and finally terminating in, these terms.*

Unequal ratios, $7 : 5,\ 11 : 9$

$$\begin{array}{llll} 9)11(1 & & 1, \quad 4, & 2 \\ \underline{9} & & & 1 : 1 \\ 2)9(4 & \left.\begin{matrix}1\\1\end{matrix}\right\}\times 4 + & \left.\begin{matrix}0\\1\end{matrix}\right\} & = 4 : 5 \\ \underline{8} & & & \\ 1)2(2 & \left.\begin{matrix}4\\5\end{matrix}\right\}\times 2 + & \left.\begin{matrix}1\\1\end{matrix}\right\} & = 9 : 11 \\ 2 & & & \end{array}$$

$$\left.\begin{matrix}7\\11\end{matrix}\right\}9 \quad \begin{matrix}63\\99\end{matrix} \quad \left.\begin{matrix}5\\9\end{matrix}\right\}11 \quad \begin{matrix}55\\99\end{matrix} \quad \left\{\begin{matrix}\textit{Here } 63 \gt r\ 55 \\ \textit{And } 99 = not \gt r\ 99\end{matrix}\right.$$

$\textit{And} \because 7 : 5 \gt r\ 11 : 9$

Unequal ratios, 7 : 5, 11 : 9

$$\left.\begin{matrix}7\\11\end{matrix}\right\}4\quad\begin{matrix}28\\44\end{matrix}\quad\left.\begin{matrix}5\\9\end{matrix}\right\}5\quad\begin{matrix}25\\45\end{matrix}\quad\left\{\begin{matrix}\textit{Here } 28 > r\ 25\\\textit{And } 44 > s \textit{ not} > r\ 45\end{matrix}\right.$$

And ∵ 7 : 5 > r 11 : 9

Unequal ratios, 9 : 11, 5 : 7

$$\left.\begin{matrix}9\\5\end{matrix}\right\}7\quad\begin{matrix}63\\35\end{matrix}\quad\left.\begin{matrix}11\\7\end{matrix}\right\}5\quad\begin{matrix}55\\35\end{matrix}\quad\left\{\begin{matrix}\textit{Here } 63 > r\ 55\\\textit{And } 35 = \textit{not} > r\ 35\end{matrix}\right.$$

And ∵ 9 : 11 > r 5 : 7 *.

DEFINITION IX.

Proportion or analogy consists of three terms at least, in which case the same magnitude is a consequent of one ratio, and an antecedent of the other; and such ratios are called *continued* ratios; or three such magnitudes are called *continued* or *continual* proportionals.

DEFINITION X, XI.

When three magnitudes are proportional, the first is said to have to the *third* the duplicate ratio of that which it has to the *second*; and when four magnitudes are in continued proportion, the *first* is said to have to the *fourth* the triplicate ratio of that which it has to the *second*; and so on quadruplicate, &c. increasing the denomination still by unity, in any number of proportionals.

That is, in any number of continued equal ratios; the ratio of the first antecedent and last consequent is *multiplicate* of the first ratio (or of any one of the equal ratios), by the *number* denoting the number of equal ratios in the series.

Suppose the ratio A : B = B : C = C : D = D : E = E : F, &c. Then when A : B = B : C, the ratio A : C is said to be to the duplicate of the ratio A : B. When A : B = B : C = C : D, the ratio A : D is said to be triplicate of the ratio A : B. When A : B = B : C = C : D = D : E, the ratio A : E is said to be quadruplicate of the ratio A : B. When A : B = B : C = C : D = D : E = E : F, the ratio A : F is said to be quintuplicate of the ratio A : B, &c. &c.

This definition is the addition *of continued equal ratios.*

DEFINITION A.

When there are any number of magnitudes of the same kind, the first is said to have to the last of them the ratio compounded of the ratio which the first has to the second, and of the ratio which the second has to the third, and of the ratio which the third has to the fourth; and so on unto the last magnitude.

For

* *Vid. Analysis, Def. XVIII. Corol. V. Scholium.*

For example, if A, B, C, D, be four magnitudes of the same kind, the first, A, is said to have to the last, D, the ratio compounded of the ratio A to B, and of the ratio B to C, and of the ratio C to D; or, the ratio of A to D is said to be compounded of the ratios A to B, B to C, and C to D. And if A has to B the same ratio which E has to F, and B to C the same ratio that G has to H, and C to D the same ratio that K has to L; then by this definition A is said to have to D the ratio compounded of ratios, which are the same with the ratios of E to F, G to H, and K to L. And the same thing is to be understood when it is more briefly expressed, by saying A has to D the ratio compounded of the ratios of E to F, G to H, and K to L.

In like manner, the same things being supposed, if M has to N the same ratio which A has to D; then, for shortness' sake, M is said to have to N the ratio compounded of the ratios of E to F, G to H, and K to L.

This definition is the addition *of continued unequal ratios.*

DEFINITION XIII, XIV, XV, XVI, XVII.

When the middle terms of an analogy change places, this is called *alternation* or *permutation.* When the terms of a ratio change places, the ratio is said to be *inverted.* And when the sum or difference of both terms of a ratio is compared with either term, the comparison is said to be *jointly* or *disjointly:* But when either term of a ratio is compared with the difference of the terms, which ever is the greater, this comparison is called *conversion.*

Let there be any ratio, as $A : B$.

The ratio $A : B$, taken by inversion, is $B : A$.

The ratio $A : B$, taken jointly, is $A + B : A$ or $A + B : B$.

The ratio $A : B$, taken disjointly, is $A \backsim B : A$ or $A \backsim B : B$.

The ratio $A : B$, taken by conversion, is $A : A \backsim B$ or $B : A \backsim B$.

DEFINITION XVIII, XIX.

When there is any number of magnitudes more than two in one series, and as many others in other series, which are proportional when taken regularly two and two in each series, from the first to the last, that is, when the first magnitude is to the second of the first series as the first to the second of the other series; and as the second is to the third of the first series, so is the second to the third of the other series; and so on in order: Then are the magnitudes said to be *ordinately* proportional, or in *ordinate* proportion. And when it is inferred, that the first is to the last of the first series of magnitudes as the first to the last of the others, this inference is said to be *ex aequali*, or *ex aequo*, from equality.

The inference in this definition is the addition *of continued ratios.*

Let there be a series of magnitudes, A, B, C, D, E, F, &c. and another series of magnitudes, as a, b, c, d, e, f, &c.

And

And ſuppoſe the ratio $A : B = a : b$
$B : C = b : c$
$C : D = c : d$
$D : E = d : e$
$E : F = e : f$
&c. &c.
Then, by equality, the ratio $A : F = a : f$.

DEFINITION XX.

When there is any number of magnitudes more than two in one, and as many others in other ſeries, which are proportional when taken two and two in each ſeries in the following order, viz. When the firſt magnitude is to the ſecond of the firſt ſeries, as the laſt but one is to the laſt of the other ſeries; and as the ſecond is to the third of the firſt ſeries, ſo is the laſt but two to the laſt but one of the other ſeries; and as the third is to the fourth of the firſt ſeries, ſo is the third from the laſt to the laſt but two of the other ſeries; and ſo on in a croſs order: Then are thoſe magnitudes ſaid to be *inordinately* proportional. And when it is inferred that the firſt is to the laſt of the firſt ſeries of magnitudes, as the firſt to the laſt of the others, this inference is ſaid to be *ex aequali, in proportione perturbata, ſeu inordinata*, from equality, in perturbate or diſorderly proportion.

The inference in this definition is the addition *of continued ratios.*

Let there be a ſeries of magnitudes, A, B, C, D, E, F, &c. and another ſeries of magnitudes, a, b, c, d, e, f, &c.

And ſuppoſe the ratio $A : B = e : f$
$B : C = d : e$
$C : D = c : d$
$D : E = b : c$
$E : F = a : b$
Then, by perturbate equality, the ratio $A : F = a : f$.

AXIOMS.

AXIOM I.

Equimultiples of the ſame, or of equal magnitudes, are equal to one another.

AXIOM II.

Thoſe magnitudes of which the ſame or equal magnitudes are equimultiples, are equal to one another.

AXIOM III.

A multiple of a greater magnitude is greater than the ſame multiple of a leſs.

AXIOM IV.

That magnitude of which a multiple is greater than the ſame multiple of another, is greater than that other magnitude.

PROPOSITIONS.

PROPOSITION I.—THEOREM.

If any number of magnitudes be equimultiples of as many, each of each, what multiple ſoever any one of them is of its part, the ſame multiple ſhall all the firſt magnitudes be of all the other.

$$\begin{array}{l|c|l}
m\,A & A & m \times A \\
m\,B & B & m \times B \\
m\,C & C & m \times C \\
m\,D & D & m \times D \\
\&c. & \&c. & \&c.
\end{array}$$

$$m\,A + m\,B + m\,C + m\,D,\ \&c. = m \times \overline{A + B + C + D,\ \&c.}$$

PROPOSITION II.—THEOREM.

If the firſt magnitude be the ſame multiple of the ſecond that the third is of the fourth, and the fifth the ſame multiple of the ſecond that the ſixth is of the fourth; then ſhall the firſt together with the fifth be the ſame multiple of the ſecond, that the third together with the ſixth is of the fourth.

$$m\,A,\ A,\ m\,B,\ B,\ n\,A,\ n\,B$$

$$m\,A + n\,A = \overline{m + n} \times A$$

$$m\,B + n\,B = \overline{m + n} \times B.$$

COROLLARY.

From this it is plain, that if any number of magnitudes, $m\,A$, $n\,A$, $p\,A$, be multiples of another A, and as many $m\,B$, $n\,B$, $p\,B$, be the ſame multiples of B, each of each, the whole of the firſt is the ſame multiple of A that the whole of the laſt is of B.

$$m\,A,\ n\,A,\ p\,A$$

$$m\,B,\ n\,B,\ p\,B$$

$$m\,A + n\,A + p\,A = \overline{m + n + p} \times A$$

$$m\,B + n\,B + p\,B = \overline{m + n + p} \times B.$$

PROPOSITION III.—THEOREM.

If the firſt be the ſame multiple of the ſecond which the third is of the fourth; and if of the firſt and third there be taken equimultiples, theſe ſhall be equimultiples, the one of the ſecond, and the other of the fourth.

Hyp.

$$\left.\begin{array}{l} mA,\ A,\ mB,\ B, \\ nmA,\ nmB, \end{array}\right\} \begin{array}{l} nmA = nm\times A \\ nmB = nm\times B. \end{array}$$

PROPOSITION IV.—THEOREM.

If the first of four magnitudes have the same ratio which the third has to the fourth; then any equimultiples whatever of the first and third shall have the same ratio to any equimultiples of the second and fourth, viz. the equimultiple of the first shall have the same ratio to that of the second which the equimultiple of the third has to that of the fourth.

Suppose $A : B = C : D$; then $mA : nB = mC : nD$.

$$\left.\begin{array}{l} mA \\ mC \end{array}\right\} p \qquad \left.\begin{array}{l} nB \\ nD \end{array}\right\} q \qquad \begin{array}{l|l} pm\times A & qn\times B \\ pm\times C & qn\times D. \end{array}$$

Since by Hypothesis $A : B = C : D$ ∵ Def. V.

If $p\times mA > r = \text{or} > s\ q\times nB$; then $p\times mC > r = \text{or} > s\ q\times nB$;

∵ Converse, Def. V. $mA : nB = mC : nD$.

COROLLARY.

Likewise, if the first has the same ratio to the second which the third has to the fourth; then also any equimultiples whatever of the first and third have the same ratio to the second and fourth; and, in like manner, the first and the third have the same ratio to any equimultiples whatever of the second and fourth.

Suppose $A : B = C : D$; then $mA : B = mC : D$,

And $A : nB = C : nD$.

$A : B = C : D$; then $mA : B = mC : D$.

$$\left.\begin{array}{l} mA \\ mC \end{array}\right\} p \qquad \left.\begin{array}{l} B \\ D \end{array}\right\} q.\ \text{Def. V.} \left\}\begin{array}{l} \text{If } pmA > r = \text{or} > s\ qB \\ \text{Then } pmC > r = \text{or} > s\ qD \end{array}\right.$$

∵ Def. V. $mA : B = mC : D$.

$A : B = C : D$; then $A : nB = C : nD$.

$$\left.\begin{array}{l} A \\ C \end{array}\right\} p \qquad \left.\begin{array}{l} nB \\ nD \end{array}\right\} q.\ \text{Def. V.} \left\{\begin{array}{l} \text{If } pA > r = \text{or} > s\ qnB \\ \text{Then } pC > r = \text{or} > s\ qnD \end{array}\right.$$

∵ Def. V. $A : nB = C : nD$.

PROPOSITION V.—THEOREM.

If one magnitude be the same multiple of another, which a magnitude taken from the first is of a magnitude taken from the other, the remainder shall be the same multiple of the remainder that the whole is of the whole.

$$\left.\begin{array}{r|c|l} mA + mB & A + B & m\times\overline{A+B} \\ mA & A & m\times A \\ \hline mB & B & m\times B \end{array}\right\} \text{or} \left\{\begin{array}{r|c|l} mA & A & m\times A \\ mB & B & m\times B \\ \hline mA - mB & A - B & m\times\overline{A-B} \end{array}\right.$$

PROPOSITION VI.—THEOREM.

If two magnitudes be equimultiples of two others, and if equimultiples of theſe be taken from the firſt two, the remainders are either equal to theſe others, or equimultiples of them.

$$\begin{array}{c|c|c|c|l} mA & A & nA & mA-nA=\overline{m-n}\times A & \text{If } m-n=1\text{, then } \overline{m-n}\times A=A \\ mB & B & nB & mB-nB=\overline{m-n}\times B & \text{And } \overline{m-n}\times B=B. \end{array}$$

When $m-n=1$, if m be even then n muſt be odd.

PROPOSITION A.—THEOREM.

If the firſt of four magnitudes has to the ſecond the ſame ratio which the third has to the fourth; then if the firſt be greater than the ſecond, the third is alſo greater than the fourth; and if equal, *equal*; if leſs, *leſs*.

$$A : B = C : D$$

$$\left.\begin{array}{c} A \\ C \end{array}\right\} m \quad \left.\begin{array}{c} B \\ D \end{array}\right\} m \qquad \begin{array}{c|c} mA & mB \\ mC & mD \end{array} \quad \because \text{Def. V.}$$

If $mA > r = \text{or} > s\ mB$, then $mC > r = \text{or} > s\ mD$ $\because$ (dividing by m), if $A > r = \text{or} > s\ B$, then $C > r = \text{or} > s\ D$.

PROPOSITION B.—THEOREM.

If four magnitudes are proportionals, they are proportionals alſo when taken inverſely.

Suppoſe $A : B = C : D$; then by inverſion, $B : A = D : C$.

$$\left.\begin{array}{c} A \\ C \end{array}\right\} m \quad \left.\begin{array}{c} B \\ D \end{array}\right\} n \text{ Def. V. } \begin{array}{l} mA > r = \text{or} > s\ nB \\ mC > r = \text{or} > s\ nD. \end{array}$$

$$\text{Or}\left\{\begin{array}{l} nB > r = \text{or} > s\ mA \\ nD > r = \text{or} > s\ mC \end{array}\right. \quad \because \text{Def. V. } B : A = D : C.$$

PROPOSITION C.—THEOREM.

If the firſt be the ſame multiple of the ſecond, or the ſame part of it that the third is of the fourth, the firſt is to the ſecond as the third is to the fourth.

Caſe I. $\left.\begin{array}{l} A = pB \\ C = pD \end{array}\right\}$ Then $A : B = C : D$.

$$\left.\begin{array}{c} A \\ C \end{array}\right\} m \quad \left.\begin{array}{c} B \\ D \end{array}\right\} n \quad \left.\begin{array}{c} mA \\ mC \end{array}\right\} = \left\{\begin{array}{l} mpB \\ mpD \end{array}\right. \quad \begin{array}{c|c} mpB & nB \\ mpD & nD. \end{array}$$

If $m > r = \text{or} > s\ n$; then $mB > r = \text{or} > s\ nB$, and $mD > r = \text{or} > s\ nD$.

As

$$\left.\begin{array}{l}\text{As also } mp\,B > r = \text{or} > s\,n\,B \\ \text{And } mp\,D > r = \text{or} > s\,n\,D\end{array}\right\} \text{or} \begin{array}{l} m\,A > r = \text{or} > s\,n\,B \\ m\,C > r = \text{or} > s\,n\,D\end{array} \because \text{Def. V.}$$

$$A : B = C : D.$$

Case II. $\left.\begin{array}{l} A = \frac{1}{p}B \\ C = \frac{1}{p}D \end{array}\right\}$ Then $A : B = C : D$.

$$\left.\begin{array}{l}\frac{1}{p}B = A \\ \frac{1}{p}D = C\end{array}\right\} \text{Hyp. or} \left.\begin{array}{l} B = p\,A \\ D = p\,C\end{array}\right\} \left\{\begin{array}{l}\because \text{Case I. } B : A = D : C\text{; and} \\ \text{inversely, } A : B = C : D \text{ (Prop. B.)}\end{array}\right.$$

PROPOSITION D.—THEOREM.

If the first be to the second as the third to the fourth, and if the first be a multiple or part of the second, the third is the same multiple, or the same part of the fourth.

Case I. Suppose $A : B = C : D$, and that $A = m\,B$; then $C = m\,D$.
For, take $E = A$ ($= m\,B$) and $F = m\,D$.
Since $A : B = C : D$, and $E = m\,B$; as also $F = m\,D$ (Hyp.)
$\because$ $A : E = C : F$ (Prop. IV. Corol.), but $A = E$ Hyp. $\because$ $C = F$ (Prop. A); that is, $A = m\,B$, and $\because$ $C = m\,D$.

Case II. Suppose $A : B = C$ D, and that $A = \frac{1}{m}B$; then $C = \frac{1}{m}D$.
Since $A : B = C : D$ (Hyp.) $\because$ Inversely, $B : A = D : C$ (Prop. B); but $A = \frac{1}{m}B$ (Hyp.) or $m\,A = B$, and $\because$ (Case I.) $D = m\,C$ or $C = \frac{1}{m}$ D; that is, $A = \frac{1}{m}B$, and $\because$ $C = \frac{1}{m}D$.

COROLLARY.

In any analogy the first term has as much magnitude when compared with the second, as the third when compared with the fourth; and, therefore, if the first term be greater, equal, or less than the second, the third term will be equally greater, equal, or greatly less than the fourth.

PROPOSITION VII.—THEOREM.

Equal magnitudes have the same ratio to the same magnitude, and the same has the same ratio to equal magnitudes.

Case I. Suppose $A = B$, and that C is any other magnitude of the same kind; then $A : C = B : C$.

$$\left.\begin{array}{l}A \\ B\end{array}\right\} m \quad \left.\begin{array}{l}C \\ C\end{array}\right\} n \quad \left.\begin{array}{l}A = B \\ m\,A = m\,B\end{array}\right\} \text{Hyp.} \because$$

$$\left.\begin{array}{l}\text{If } m\,A > r = \text{or} > s\,n\,C \\ \text{Then } m\,B > r = \text{or} > s\,n\,C\end{array}\right\} \text{And} \because \text{Def. V. } A : C = B : C.$$

Case

Case II. $C : A = C : B$

$$\left.\begin{matrix} C \\ C \end{matrix}\right\} n \quad \left.\begin{matrix} A \\ B \end{matrix}\right\} m \quad \left.\begin{matrix} A = B \\ m\,A = m\,B \end{matrix}\right\} \begin{matrix} \text{Hyp.} \because \text{ If } n\,C > r = \text{or} > s\ m\,A \\ \text{Then } n\,C > r = \text{or} > s\ m\,B\,; \end{matrix}$$

And consequently, Def. V. $C : A = C : B$.

PROPOSITION VIII.—THEOREM.

Of unequal magnitudes the greater has a greater ratio to the same than the less has, and the same magnitude has a greater ratio to the less than it has to the greater.

$$\text{Hyp.} \left\{\begin{matrix} A > r\ B \\ C \text{ any third magnitude} \\ \text{of the same kind :} \end{matrix}\right\} \text{Then} \left\{\begin{matrix} A : C > r\ B : C \\ C : B > r\ C : A. \end{matrix}\right.$$

Since $A > r\ B$ (Hyp.), the difference of A, B is $A - B$. Then, if the magnitude *which* is *not* the greater of the two $A - B$, B be not *less* than C, take $2\,\overline{A-B}$, $2\,B$, the doubles of $A - B$, B; but if that which is *not* the *greater* of the two $A - B$, B be less than C, it may be multiplied so as to become greater than C, whether it be $A - B$ or B. Let it be multiplied, therefore, until it become *greater* than C; and let the other be multiplied *as often*. Let $m\,\overline{A-B}$, $m\,B$, be these equimultiples, which are, therefore, each *greater* than C: And in every one of the cases, let $q\,C$ be that multiple of C which *first* becomes greater than $m\,B$ and $p\,C$ $(= \overline{q-1} \times C)$, the multiple of C which is *next less* than $q\,C$.

Because $q\,C$ is the multiple of C, which is the *first* that becomes $> r\ m\,B$; the next preceding multiple $p\,C$ is *not* $> r\ m\,B$; that is, $m\,B$ is *not* $> s\ p\,C$. And $m.\overline{A-B}$, $m\,B$, being equimultiples of the magnitudes $A - B$, B; the sum $(m\,\overline{A-B} + m\,B =)$ $m\,A$ and $m\,B$, are the same equimultiples of A B (Prop. I.), as is evident also by the notation.

But it was shown that $m\,B$ *not* $> s\ p\,C$, and $m\,\overline{A-B} > r\ C$ $\because$ the whole $(m\,B + m\,\overline{A-B} =)$ $m\,A > r$ $(p\,C + C = \overline{p+1} \times C =)$ $q\,C$ (Hyp.); but as shown, $q\,C > r\ m\,B$, or $m\,B$ not $> r\ q\,C$. Thus:

$$\begin{matrix} A,\ C,\ B,\ C \\ \left.\begin{matrix} A \\ B \end{matrix}\right\} m \quad \left.\begin{matrix} C \\ C \end{matrix}\right\} q \end{matrix} \left\{\begin{matrix} m\,A > r\ q\,C \\ m\,B \text{ not} > r\ q\,C \end{matrix}\right\} \begin{matrix} \because \text{Def. VII.} \\ A : C > r\ B : C. \end{matrix}$$

Also, $C : B > r\ C : A$. For having stated the same multiples, &c. as above, it may be shown, in like manner, that $q\,C > r\ m\,B$, and *not* $> r\ m\,A$.

$$\begin{matrix} C,\ B,\ C,\ A \\ \left.\begin{matrix} C \\ C \end{matrix}\right\} q \quad \left.\begin{matrix} B \\ A \end{matrix}\right\} m \end{matrix} \left\{\begin{matrix} q\,C > r\ m\,B \\ q\,C \text{ not} > r\ m\,A \end{matrix}\right\} \begin{matrix} \because \text{Def. VII.} \\ C : B > r\ C : A. \end{matrix}$$

Otherwise*.

$$\left.\begin{matrix} \overline{A+B} > r\ A \\ C \end{matrix}\right\} \text{Then} \left\{\begin{matrix} \overline{A+B} : C > r\ A : C \\ C : A > r\ C : \overline{A+B}. \end{matrix}\right.$$

Take

* *Mr Playfair's Geometry, Book V. p.* 142.

Take m, n, multipliers, such that $m\,A \triangleright r\,C$, $m\,B \triangleright r\,C$, and $n\,C$ the least multiple of C, exceeding $m\,A+m\,B$ $(= m \times \overline{A+B})$.

$$\left.\begin{array}{l}\text{Hyp. } n\,C \triangleright r\ m\,A+m\,B \\ \therefore \overline{n-1}\ C \triangleright s\ m\,A+m\,B\end{array}\right\}\ \text{or}\ \left\{m\,A+m\,B \triangleright r\ \overline{n-1}.C.\right.$$

$$\left.\begin{array}{l}\text{Hyp. } n\,C \triangleright r\ m\,A+m\,B \\ \because (n\,C-C=)\overline{n-1}.C \triangleright s\ m\,A+\overline{m\,B-C}\end{array}\right]\ \left[\begin{array}{l}\text{Hyp. } m\,B \triangleright r\,C \\ \because \overline{n-1}\ C \triangleright r\ m\,A.\end{array}\right.$$

Thus:

$$\begin{array}{l}(m\,A+m\,B=)\ m.\overline{A+B} \triangleright r\ \overline{n-1}.C \\ \quad m\,A\ (\triangleright s \text{ and) not } \triangleright r\ \overline{n-1}\ C \\ \because \text{Def. VII. } \overline{A+B} : C \triangleright r\ A : C. \\ \text{Also, as } \overline{n-1}\ C \triangleright r\ m\,A \\ \qquad \overline{n-1}\ C\ (\triangleright s \text{ and) not } \triangleright r\ m\,\overline{A+B} \\ \because \text{Def. VII. } C : A \triangleright r\ C : \overline{A+B}.\end{array}$$

PROPOSITION IX.

Magnitudes which have the same ratio to the same magnitude, are equal to one another; and those to which the same magnitude has the same ratio, are equal to one another.

$$\text{Hyp.} \left[\begin{array}{l}A : C = B : C \\ C : B = C : A\end{array}\right] \text{Then } A = B.$$

If A, B are not equal, they must be *unequal*, and one *necessarily* greater than the other, which suppose to be A. Then since $A \triangleright r\ B$, and C a third magnitude of the same kind: there are (Prop. VIII.) *some* equimultiples $m\,A$, $m\,B$ of A, B, and some equimultiple $q\,C$ of C, such that $m\,A \triangleright r\ q\,C$ and $m\,B$ *not* $\triangleright r\ q\,C$, or $q\,C \triangleright r\ m\,B$ and *not* $\triangleright r\ m\,A$. But, because $A : C = B : C$ and $C : B = C : A$ (Hyp.) $\because$ Def. V. if $m\,A \triangleright r\ q\,C$, then $m\,B \triangleright r\ q\,C$; or if $q\,C \triangleright r\ m\,B$, then $q\,C \triangleright r\ m\,A$; but $m\,B$ *not* $\triangleright r\ q\,C$, and $q\,C$ *not* $\triangleright r\ m\,A$, which are impossible. Therefore, since the supposed inequality of A, B involves an *impossibility* in both cases, the magnitudes of A, B are not unequal, that is, they are equal.

PROPOSITION X.

That magnitude which has a greater ratio than another has unto the same magnitude, is the greater of the two; and that magnitude to which the same has a greater ratio than it has unto another magnitude, is the less of the two.

$$\text{Hyp.} \left[\begin{array}{l}A : C \triangleright r\ B : C \\ C : B \triangleright r\ C : A\end{array}\right. \left[\text{Then } A \triangleright r\ B.\right.$$

Because $A : C \triangleright r\ B : C$, and $C : B \triangleright r\ C : A$ (Hyp.): there are *some* (Def. VII.) equimultiples $m\,A$, $m\,B$ of A B, and *some* multiple $q\,C$ of C, such that $m\,A \triangleright r\ q\,C$ and $m\,B$ *not* $\triangleright r\ q\,C$, or $q\,C \triangleright r\ m\,B$ and *not* $\triangleright r\ m\,A$. Therefore, since $m\,A \triangleright r\ m\,B$ in both cases, it is $A \triangleright r\ B$ (Axiom IV.).

PROPOSITION XI.

Ratios that are the ſame to the ſame ratio, are the ſame to one another.

$$\text{Hyp.} \left[\begin{array}{l} A : B = C : D \\ C : D = E : F \end{array}\right] \text{Then } A : B = E : F.$$

$$\left.\begin{array}{c} A \\ C \\ E \end{array}\right\} m \quad \left.\begin{array}{c} B \\ D \\ F \end{array}\right\} n \quad \left.\begin{array}{c} A : B = C : D \text{ Hyp.} \\ \because \text{Def. V.} \\ mA > r = \text{or} > s\ nB \\ mC > r = \text{or} > s\ nD \end{array}\right\} \left\{\begin{array}{c} C : D = E : F \text{ Hyp.} \\ \because \text{Def. V.} \\ mC > r = \text{or} > s\ nD \\ mE > r = \text{or} > s\ nF. \end{array}\right\}$$

$$\begin{array}{l} \text{Thus: when } mC > r = \text{or} > s\ nD \\ \left.\begin{array}{l} mA > r = > s\ nB \\ mE > r = > s\ nF \end{array}\right\} \begin{array}{l} \because \text{Def. V.} \\ A : B = E : F. \end{array} \end{array}$$

COROLLARY.

Ratios that are the ſame to the ſame ratios, are the ſame to one another.

PROPOSITION XII.—THEOREM.

If any number of magnitudes be proportionals, as one of the antecedents is to its conſequent, ſo ſhall all the antecedents taken together be to all the conſequents.

$$\text{Hyp.} \mid A : B = C : D = E : F, \text{ \&c.} \mid \text{Then } A : B = A + C + E, \text{ \&c.} : B + D + F, \text{ \&c.}$$

$$A : B = C : D = E : F, \text{ \&c. (Hyp.)}$$

$$\left.\begin{array}{c} A \\ C \\ E \\ \text{\&c.} \end{array}\right\} m \qquad \left.\begin{array}{c} B \\ D \\ F \\ \text{\&c.} \end{array}\right\} n$$

And by Def. V.

$$\text{If } mA > r = \text{or} > s\ nB$$

$$\begin{array}{l} \text{Then } mC > r = \text{or} > s\ nD \\ \quad mE > r = \text{or} > s\ nF \\ \quad \text{\&c.} \qquad \text{\&c.} \end{array}$$

$$\text{And conſequently, } mA + mC + mE \text{ \&c.} > r = \text{or} > s\ nB + nD + nF \text{ \&c.},$$

$$\text{Or } m \times \overline{A + C + E \text{ \&c.}} > r = \text{or} > s\ n \times \overline{B + D + F \text{ \&c.}}$$

Therefore, by Def. V.

$$A : B\ (= C : D = E : F \text{ \&c.}) = A + C + E \text{ \&c.} : B + D + F \text{ \&c.}$$

PROPOSITION XIII.—THEOREM.

If the firſt has to the ſecond the ſame ratio which the third has to the fourth, but the third has to the fourth a greater ratio than the fifth has to the ſixth; the firſt ſhall alſo have to the ſecond a greater ratio than the fifth has to the ſixth.

$$\text{Hyp.} \left\{\begin{array}{l} A : B = C : D \\ C : D \triangleright r\, E : F \end{array}\right\} \text{Then } A : B \triangleright r\, E : F.$$

Since $C : D \triangleright r\, E : F$ (Hyp.), there are (Def. VII.) *ſome* equimultiples $m\,C$, $m\,E$, of C, E, and *ſome* equimultiples of $n\,D$, $n\,F$, of D, F, ſuch that $m\,C \triangleright r\, n\,D$, and $m\,E$ *not* $\not\triangleright r\, n\,F$.

$$\left.\begin{array}{l} \text{Alſo, ſince } A : B = C : D, \text{ Hyp.} \\ \left.\begin{array}{l} A \\ C \end{array}\right\} m \quad \left.\begin{array}{l} B \\ D \end{array}\right\} n \end{array}\right\} \begin{array}{l} \because \text{ Def. V.} \\ \text{If } m\,A \triangleright r = \text{or} \triangleright s\, n\,B \\ \text{Then } m\,C \triangleright r = \text{or} \triangleright s\, n\,D. \end{array}$$

$$\left.\begin{array}{l} \text{By ſuppoſition, } m\,C \triangleright r\, n\,D \\ \because \text{ Def. V. } \; m\,A \triangleright r\, n\,B \end{array}\right\} \left.\begin{array}{l} \text{Thus } m\,A \triangleright r\, n\,B \\ m\,E \text{ not} \triangleright r\, n\,F \end{array}\right\} \begin{array}{l} \because \text{ Def. VII.} \\ A : B \triangleright r\, E : F. \end{array}$$

COROLLARY I.

If the firſt has a greater ratio to the ſecond than the third has to the fourth, but the third has the ſame ratio to the fourth which the fifth has to the ſixth; it may be demonſtrated, in like manner, that the firſt has a greater ratio to the ſecond than the fifth has to the ſixth.

$$\text{Hyp.} \left\{\begin{array}{l} A : B \triangleright r\, C : D \\ C : D = E : F \end{array}\right\} \text{Then } A : B \triangleright r : E : F.$$

COROLLARY II.

If the firſt has the ſame ratio to the ſecond which the third has to the fourth, but the third has a leſs ratio to the fourth than the fifth has to the ſixth; then the firſt has a leſs ratio to the ſecond than the fifth has to the ſixth.

$$\text{Hyp.} \left\{\begin{array}{l} A : B = C : D \\ C : D \triangleright s\, E : F \end{array}\right\} \text{Then } A : B \triangleright s\, E : F.$$

COROLLARY III.

If the firſt has a greater or leſs ratio to the ſecond than the third has to the fourth, and the third has a greater or leſs ratio to the fourth than the fifth has to the ſixth; then the firſt has a greater or leſs ratio to the ſecond than the fifth has to the ſixth.

$$\text{Hyp.} \left\{\begin{array}{l} A : B \triangleright r \text{ or} \triangleright s\, C : D \\ C : D \triangleright r \text{ or} \triangleright s\, E : F \end{array}\right\} \text{Then } A : B \triangleright r \text{ or} \triangleright s\, E : F.$$

PROPOSITION XIV.—THEOREM.

[illegible] ratio which the third has to the fourth; then, if the first be greater than the third, the second shall be greater than the fourth; if equal, equal; and if less, less.

If $A > r = \text{or} > s\ C$
Then $B > r = \text{or} > s\ D$.

II. $\because A = C$, Hyp.	III. $A > s\,C$, Hyp.
$\therefore A : B = C : D$, Hyp.	$\because C > r\,A$
Or $A : B = A : D$.	$C : D = A : B$, Hyp.
$\because B = D$, Prop. IX.	$\because D > r\,B$, Case I.
	Or $B > s\,D$.

PROPOSITION XV.—THEOREM.

Magnitudes have the same ratio to one another which their equimultiples have.

$A : B = mA : mB$, m being *any* number.
$A + A + A + A$, &c. to m terms $= mA$.
$B + B + B + B$, &c. to m terms $= mB$.
$A = A$, $B = B$.
$A : B = A : B$, &c. to m terms.
$\therefore A : B = mA : mB$ (Prop. XII.).

COROLLARY.

Magnitudes have the same ratio to one another which their equisubmultiples have.

$$A : B = \tfrac{1}{m} A : \tfrac{1}{m} B.$$

$$\text{Suppose} \left\{ \begin{array}{l} C = \tfrac{1}{m} A \\ D = \tfrac{1}{m} B \end{array} \right\} \text{Then} \left\{ \begin{array}{l} mC = A \\ mD = B \end{array} \right\}$$

$$mC : mD = C : D \text{ (Prop.)}$$

$$\text{Or } A : B = \tfrac{1}{m} A : \tfrac{1}{m} B.$$

SCHOLIUM.

This Proposition, though not given by Euclid, is equally useful with the Pro-

PROPOSITION XVI.—THEOREM.

[illegible] the same kind be proportionals, they shall also be [illegible] alternately.

Hyp.

Hyp. | $A : B = C : D$ | *Alternately*, $A : C = B : D$.

$$\left.\begin{array}{c}A\\B\end{array}\right\} m \qquad \left.\begin{array}{c}C\\D\end{array}\right\} n$$

$A : B = mA : mB$, Prop. XV.
$A : B = C : D$, Hyp.
$\therefore C : D = mA : mB$, Prop. XI.

$C : D = nC : nD$, Prop. XV.
$\therefore mA : mB = nC : nD$, Prop. XI.
$\left.\begin{array}{l}\therefore \text{If } mA > r = \text{or} > s\, nC\\ \text{Then } mB > r = \text{or} > s\, nD\end{array}\right\}$ Pr. XIV.

And $\because$ Def. V. $A : C = B : D$.

PROPOSITION XVII.—THEOREM.

If magnitudes taken jointly be proportionals, they ſhall alſo be proportionals when taken disjointly; that is, if two magnitudes together have to one of them the ſame ratio which two others have to one of theſe, the remaining one of the firſt two ſhall have to the other the ſame ratio which the remaining one of the laſt two has to the other of theſe.

Hyp. | $A+B : B = C+D : D$ | *Disjointly*, $A : B = C : D$.

$$\left.\begin{array}{c}A\\B\\C\\D\end{array}\right\} m \qquad \left.\begin{array}{c}B\\D\end{array}\right\} n$$

$$\left.\begin{array}{r}mA = m \times A\\ mB = m \times B\\ mA + mB = m \times \overline{A+B}\\ \text{But } \overline{A+B} > r\, B\\ \therefore m \times \overline{A+B} > r\, mB\end{array}\right\} \quad \left\{\begin{array}{r}mC = m \times C\\ mD = m \times D\\ mC + mD = m \times \overline{C+D}\\ \text{But } \overline{C+D} > r\, D\\ \therefore m \times \overline{C+D} > r\, mD\end{array}\right.$$

$A+B : B = C+D : D$, Hyp.

$$\left.\begin{array}{r}\therefore \text{Def. V.}\\ \text{If } m.\overline{A+B} > r = \text{or} > s\, nB\\ \text{Then } m.\overline{C+D} > r = \text{or} > s\, nD\end{array}\right\} \quad \left\{\begin{array}{l}\text{Whence, taking } mB \text{ from the former, and } mD \text{ from the latter,}\\ \text{If } mA > r = \text{or} > s\, \overline{n-m} \times B\\ \text{Then } mC > r = \text{or} > s\, \overline{n-m} \times D.\end{array}\right.$$

And $\because$ Def. V. $A : B = C : D$.

PROPOSITION XVIII.—THEOREM.

If magnitudes taken disjointly be proportionals, they ſhall alſo be proportionals when taken jointly; that is, if the firſt be to the ſecond as the third to the fourth, the firſt and ſecond together ſhall be to the ſecond as the third and fourth together to the fourth.

Hyp. | $A+B : B = C+D : D$ | Jointly, $A : B = C : D$.

$$\left.\begin{array}{c}A\\B\\C\\D\end{array}\right\} m \qquad \left.\begin{array}{c}B\\D\end{array}\right\} n$$

$$\left.\begin{array}{r} mA = m\times A \\ mB = m\times B \\ mA - mB = m\times\overline{A-B}\end{array}\right\}\quad\left\{\begin{array}{r} mC = m\times C \\ mD = m\times D \\ mC - mD = m\times\overline{C-D}\end{array}\right.$$

$A-B : B = C-D : D$, Hyp.

$$\left.\begin{array}{l} \because \text{Def. V.} \\ \text{If } m.\overline{A-B} > r = \text{or} > s\ nB \\ \text{Then } m.\overline{C-D} > r = \text{or} > s\ nD \end{array}\right\}\begin{array}{l} \text{Whence, adding } mB \text{ to the former, and} \\ \quad mD \text{ to the latter,} \\ \text{If } mA > r = \text{or} > s\ \overline{m+n}\times B \\ \text{Then } mC > r = \text{or} > s\ \overline{m+n}\times D \end{array}$$

And $\because$ Def. V, $A : B = C : D$.

PROPOSITION XIX.—THEOREM.

If a whole magnitude be to a whole as a magnitude taken from the first is to a magnitude taken from the other, the remainder shall be to the remainder as the whole to the whole.

Hyp. $A+B : C+D = B : D$ | Then $A : C = A+B : C+D$.

Since $A+B : C+D = B : D$, Hyp.

$\because$ Alternately, $A+B : B = C+D : D$, Prop. XVI.

Disjointly, $A : B = C : D$, Prop. XVII.

Alternately, $A : C = B : D$, Prop. XVI.

Hyp. $A+B : C+D = B : D$.

$\because A : C = A+B : C+D$, Prop. XI.

COROLLARY.

If the whole be to the whole as a magnitude taken from the first is to a magnitude taken from the other, the remainder likewise is to the remainder as the magnitude taken from the first to that taken from the other.

Hyp. | $A+B : C+D = B : D$ | Then $A : C = B : D$.

The fourth step of the demonstration.

PROPOSITION E.—THEOREM.

If four magnitudes be proportionals, they are also proportionals by conversion; that is, the first is to its excess above the second as the third to its excess above the fourth.

Hyp. | $A+B : B = C+D : D$ | Conversely, $A+B : A = C+D : C$.

Since $A+B : B = C+D : D$, Hyp.

$\because$ Disjointly, $A : B = C : D$, Prop. XVII.

Inversely, $B : A = D : C$, Prop. B.

Jointly, $A+B : A = C+D : C$, Prop. XVIII.

PROPOSITION

PROPOSITION XX.—THEOREM.

If there be three magnitudes, and other three, which taken two and two ordinately (Def. XVIII, XIX.), have the same ratio; if the first be greater than the third, the fourth shall be greater than the sixth; and if equal, equal; and if less, less.

$$\text{Hyp.}\left\{\begin{array}{l}A, B, C\\ D, E, F\end{array}\right\}\begin{array}{l}A : B = D : E\\ B : C = E : F\end{array}\left\}\begin{array}{l}\text{If } A > r = \text{or} > s\ C\\ \text{Then } D > r = \text{or} > s\ F.\end{array}\right.$$

I. $A > r\ C$, Hyp.
$\because A : B > r\ C : B$, Prop. VIII.
But $D : E = A : B$, Hyp.
$\because D : E > r\ C : B$, Prop. XIII.
And $B : C = E : F$, Hyp.
Or $C : B = F : E$, Prop. B.
$\because D : E > r\ F : E$, Prop. XIII. Cor.
And $\because D > r\ F$, Prop.

II. $A = C$, Hyp.
$\because A : B = C : B$, Prop. VII.

$$\left.\begin{array}{l}\text{But } A : B = D : E\\ C : B = F : E, \text{Prop. B.}\end{array}\right\}\text{Hyp.}$$

$\because D : E = F : E$, Prop. XI.
$\because D = F$, Prop. IX.

III. $A > s\ C$, Hyp.
$\because C > r\ A$
$C : B = F : E$, Case I.
$B : A = E : D$, Hyp. Prop. B.

$$\left\{\begin{array}{l}C, B, A\\ F, E, D\end{array}\right\}\left\{\begin{array}{l}C : B = F : E\\ B : A = E : D\end{array}\right\}$$

$\because F > r\ D$, Case I.
Or $D > s\ F$.

PROPOSITION XXI.—THEOREM.

If there be three magnitudes, and other three, which taken two and two *inordinately* (Def. XX.), have the same ratio; if the first magnitude be greater than the third, the fourth shall be greater than the sixth; and if equal, equal; and if less, less.

$$\text{Hyp.}\left(\begin{array}{l}A, B, C\\ D, E, F\end{array}\right)\left(\begin{array}{l}A : B = E : F\\ B : C = D : E\end{array}\right)\begin{array}{l}\text{If } A > r = \text{or} > s\ C\\ \text{Then } D > r = \text{or} > s\ F.\end{array}$$

I. $A > r\ C$, Hyp.
$\because A : B > r\ C : B$, Prop. VIII.
But $E : F = A : B$, Hyp.
$\because E : F > r\ C : B$, Prop. XIII.
And $B : C = D : E$, Hyp.
Or $C : B = E : D$, Prop. B.
$\because E : F > r\ E : D$, Prop. XIII. Cor.
And $\because F > s\ D$, Prop. X.
Or $D > r\ F$.

II. $A = C$, Hyp.
$\because A : B = C : B$, Prop. VII.
But $A : B = E : F$, Hyp.
$C : B = E : D$, Hyp. Prop. B.
$\because E : F = E : D$, Prop. XI,
And $\because D = F$, Prop. IX.

III. $A > s\ C$
$\because C > r\ A$
$\left.\begin{matrix}A, B, C\\ D, E, F\end{matrix}\right)\begin{matrix}C : B = E : D, \text{ Hyp. Prop. B.}\\ B : A = F : E, \text{ Hyp. Prop. B.}\end{matrix}$
$\because F > r\ D$, Cafe I.
Or $D > s\ F$.

PROPOSITION XXII.—THEOREM.

If there be any number of magnitudes, and as many others, which taken two and two *ordinately*, have the fame ratio; then, by equality (Def. XVIII, XIX.), the firft fhall have to the laft of the firft magnitudes the fame ratio which the firft of the others has to the laft.

$$\text{Hyp. I.}\left\{\begin{matrix}A,B,C\\ D,E,F\end{matrix}\right\}\begin{matrix}A:B = D:E\\ B:C = E:F\end{matrix}\left\{\begin{matrix}\text{Equality}\\ A:C = D:F\end{matrix}\right\}\left\{\begin{matrix}A\\ D\end{matrix}\right\}m\left.\begin{matrix}B\\ C\end{matrix}\right\}n\left.\begin{matrix}C\\ F\end{matrix}\right\}p.$$

$$\text{Hyp.}\left\{\begin{matrix}A : B = D : E\\ B : C = E : F\end{matrix}\right\}\because \text{Prop. IV.}\left\{\begin{matrix}mA : nB = mD : nE\\ nB : pC = nE : pF\end{matrix}\right\}$$

$$\text{Thus}\left\{\begin{matrix}mA,\ nB,\ pC\\ mD,\ nE,\ pF\end{matrix}\right\}\text{And}\left\{\begin{matrix}mA : nB = mD : nE\\ nB : pC = nE : pF.\end{matrix}\right.$$

$$\because \text{Prop. XX.}\left\{\begin{matrix}\text{If } mA > r = \text{or} > s\ pC\\ \text{Then } mD > r = \text{or} > s\ pF\end{matrix}\right\}\because \text{Def. V.}\left\{A:C = D:F.\right.$$

$$\text{Hyp. II.}\left\{\begin{matrix}A,\ B,\ C,\ D, \&c.\\ E,\ F,\ G,\ H, \&c.\end{matrix}\right.\left\{\begin{matrix}A : B = E : F\\ B : C = F : G\\ C : D = G : H\\ \&c.\ \&c.\end{matrix}\right\}\begin{matrix}\text{Equality.}\\ A : D = E : H.\end{matrix}$$

$$\left\{\begin{matrix}A,B,C\\ E,F,G\end{matrix}\right\}\begin{matrix}A : B = E : F, \text{Hyp.}\\ B : C = F : G, \text{Hyp.}\end{matrix}\left.\right\}\begin{matrix}\because \text{Cafe I.}\\ A : C = E : G\end{matrix}\left\{\begin{matrix}A : C = E : G\\ C : D = G : H, \text{Hyp.}\end{matrix}\right.$$

$$\left.\begin{matrix}\because \text{Cafe I.}\\ A : D = E : H\end{matrix}\right\}$$ And fo on, whatever be the number of magnitudes.

PROPOSITION XXIII.—THEOREM.

If there be any number of magnitudes, and as many others, which taken two and two *inordinately*, have the fame ratio; then, by perturberate equality (Def. XX.), the firft fhall have to the laft of the firft magnitudes the fame ratio which the firft of the others has to the laft.

$$\text{Hyp. I.}\left(\begin{matrix}A,\ B,\ C\\ D,\ E,\ F\end{matrix}\right)\begin{matrix}A : B = E : F\\ B : C = D : E\end{matrix}\left(\begin{matrix}\text{Perturberate equality.}\\ A : C = D : F.\end{matrix}\right.$$

$$\left.\begin{array}{l}A\\B\\D\end{array}\right\}m \quad \left.\begin{array}{l}C\\E\\F\end{array}\right\}n \quad \text{Prop. XV.}\left(\begin{array}{l}A:B=mA:mB\\E:F=nE:nF\end{array}\right)\begin{array}{l}\because \text{Prop. XI.}\\mA:mB=nE:nF.\end{array}$$

$$\text{Hyp.}\left(B:C=D:E\right)\begin{array}{l}\because \text{Prop. IV.}\\mB:nC=mD:nE.\end{array}$$

$$\left(\begin{array}{l}mA,\ mB,\ nC\\mD,\ nE,\ nF\end{array}\right)\begin{array}{l}mA:mB=nE:nF\\mB:nC=mD:nE\end{array}\left(\begin{array}{l}\text{If } mA > = \text{or} < nC\\\text{Then } mD > = \text{or} < nF\end{array}\right)\begin{array}{l}\text{Prop.}\\\text{XXI.}\end{array}$$

$$\because \text{Def. V.}\ A:C=D:F.$$

$$\text{Hyp. II.}\left(\begin{array}{l}A,\ B,\ C,\ D,\ \&c.\\E,\ F,\ G,\ H,\ \&c.\end{array}\right)\left.\begin{array}{c}A:B=G:H\\B:C=F:G\\C:D=E:F\\\&c.\ \&c.\end{array}\right\}\begin{array}{c}\text{Perturberate equality.}\\A:D=E:H.\end{array}$$

$$\left(\begin{array}{l}A,\ B,\ C\\F,\ G,\ H\end{array}\right)\begin{array}{l}A:B=G:H,\ \text{Hyp.}\\B:C=F:G,\ \text{Hyp.}\end{array}\left(\begin{array}{l}\because \text{Case I.}\\A:C=F:H\end{array}\right.\left(\begin{array}{l}A:C=F:H,\ \text{Case I.}\\C:D=E:F,\ \text{Hyp.}\end{array}\right.$$

$$\left.\begin{array}{l}\because \text{Case I.}\\A:D=E:H\end{array}\right)\text{And so on, whatever be the number of magnitudes.}$$

SCHOLIUM.

From this Proposition and Proposition XXI, as also from Proposition V and Proposition XX, it follows directly, that ratios compounded of the same or of equal ratios, are the same with one another; as particularly demonstrated by Dr Simson, in Propositions F, G, H, I, and K, following Proposition XXV, which are nothing more than simple inferences or corollaries from this Proposition and Proposition XXI.

PROPOSITION XXIV.—THEOREM.

If the first has to the second the same ratio which the third has to the fourth, and the fifth to the second the same ratio which the sixth has to the fourth; the first and fifth together shall have to the second the same ratio which the third and sixth together have to the fourth.

$$\text{Hyp.}\left(\begin{array}{l}A:B=C:D\\E:B=F:D\end{array}\right)\begin{array}{c}\text{Then}\\A+E:B=C+F:D.\end{array}$$

$$\begin{array}{l}E:B=F:D,\ \text{Hyp.}\\\because B:E=D:F,\ \text{Prop. B.}\end{array}\left[\begin{array}{l}A:B=C:D,\ \text{Hyp.}\\B:E=D:F,\ \text{Hyp. Prop. B.}\end{array}\right.$$

$$\left.\begin{array}{r}\because \text{Equality, Prop. XXII.}\\A:E=C:F\end{array}\right]\begin{array}{l}\because \text{Jointly, Prop. XVIII.}\\A+E:E=C+F:F.\end{array}$$

$$\left.\begin{array}{l}\text{Thus } E:B=F:D,\ \text{Hyp.}\\\text{And } A+E:E=C+F:F\end{array}\right]\begin{array}{l}\because \text{Equality, Prop. XXII.}\\A+E:B=C+F:D.\end{array}$$

COROLLARY

COROLLARY I.

If the ſame hypotheſis be made as in the Propoſition, the exceſs of the firſt and fifth ſhall be to the ſecond as the exceſs of the third and ſixth to the fourth.

That is, $A \backsim E : B = C \backsim F : D$.

The demonſtration of this is the ſame with that of the Propoſition, if terms be taken disjointly inſtead of jointly.

COROLLARY II.

The Propoſition holds true of two ranks of magnitudes, whatever be their numbers, of which each of the firſt rank has to a ſecond magnitude the ſame ratio that the correſponding one of the ſecond rank has to a fourth magnitude, as is manifeſt.

PROPOSITION XXV.—THEOREM.

If four magnitudes are proportionals, the greateſt and leaſt of them together are greater than the other two together.

$$\text{Hyp.}\left\{\begin{array}{l} G+H : K+L = H : L; \\ \text{where } G+H \text{ is the greateſt,} \\ \text{and } \because L \text{ the leaſt, Prop. A, XIV.} \end{array}\right\}\begin{array}{l}\text{Then} \\ G+H+L > r\,\overline{K+L}+H.\end{array}$$

Since $G+H : K+L = H : L$, Hyp.
$\because G : K = G+H : K+L$, Prop. XIX.

But $\overline{G+H} > r\, \overline{K+L}$, Hyp.
$\because G > r\, K$, Prop. A.

And $\because G+\overline{H+L} > r\, K+\overline{H+L}$, Ax. IV. e. 1.

That is, $\overline{G+H}+L > r\, \overline{K+L}+H$.

Otherwiſe.

$$\text{Hyp.}\left\{\begin{array}{l} A : B = C : D. \\ A \text{ greateſt, } D \text{ leaſt.} \end{array}\right\}\begin{array}{l}\text{Then} \\ \overline{A+D} > r\, \overline{B+C}.\end{array}$$

Since $A : B = C : D$, Hyp.
$\because A-C : B-D = A : B$, Prop. XIX. Corol.
But $A > r\, B$, Hyp.
$\because A-C > r\, B-D$, Prop. A.
And $C+D = C+D$
$\because \overline{A+D} > r\, \overline{B+C}$, Ax. IV. e. 1.

PROPOSITION F.—THEOREM.

Ratios which are compounded of the ſame ratios, are the ſame with one another.

$$\text{Hyp.}\left\{\begin{array}{l}A,B,C\\ D,E,F\end{array}\right.\left\{\begin{array}{l}A:B=D:E\\ B:C=E:F\end{array}\right\}\begin{array}{c}\text{Then}\\ A:C=D:F\end{array}\left\{\begin{array}{l}A:B=E:F\\ B:C=D:E\end{array}\right\}\begin{array}{c}\text{Then}\\ A:C=D:F.\end{array}$$

$$\times\ \text{Def.}$$

$$\text{Hyp.}\left\{\begin{array}{l}A,B,C\\ D,E,F\end{array}\right.\left\{\begin{array}{l}A:B=D:E\\ B:C=E:F\end{array}\right.\left\{\begin{array}{l}\because\text{Prop. XXII.}\\ A:C=D:F\end{array}\right\}\begin{array}{l}A:C=\overline{A:B}+\overline{B:C}\\ D:F=\overline{D:E}+\overline{E:F}.\end{array}$$

$$\times\ \text{Def.}$$

$$\text{Hyp.}\left\{\begin{array}{l}A,B,C\\ D,E,F\end{array}\right\}\begin{array}{l}A:B=E:F\\ B:C=D:E\end{array}\left\{\begin{array}{l}\because\text{Prop. XXIII.}\\ A:C=D:F\end{array}\right\}\left\{\begin{array}{l}A:C=\overline{A:B}+\overline{B:C}\\ D:F=\overline{E:F}+\overline{D:E}.\end{array}\right.$$

PROPOSITION G.—THEOREM.

If ſeveral ratios be the ſame with ſeveral ratios, each to each, the ratio which is compounded of ratios which are the ſame with the firſt ratios, each to each, is the ſame with the ratio compounded of ratios which are the ſame with the other ratios, each to each.

$$\text{Hyp.}\left\{\begin{array}{l}A,B,C,D\\ E,F,G,H\end{array}\right\}\left\{\begin{array}{l}K,L,M\\ N,O,P\end{array}\right\}\begin{array}{l}A:B=E:F,C:D=G:H\\ A:B=K:L,C:D=L:M\end{array}\left\{\begin{array}{l}E:F=N:O\\ G:H=O:P\end{array}\right.$$

$$\text{Then}\ |\ K:M=N:P.\ |$$

$$\begin{array}{l}K,L,M\\ N,O,P\end{array}\left\{\begin{array}{l}K:L(=A:B=E:F)=N:O,\ \text{Hyp.}\\ L:M(=C:D=G:H)=O:P,\ \text{Hyp.}\end{array}\right\}\begin{array}{l}\because\text{Prop. XXII.}\\ K:M=N:P.\end{array}$$

$$\left\{\begin{array}{l}K:M=\overline{K:L}+\overline{L:M},\text{Def. X.}=\overline{A:B}+\overline{C:D}\\ N:P=\overline{N:O}+\overline{O:P},\text{Def. X.}=\overline{E:F}+\overline{G:H}\end{array}\right\}\text{Hyp.}$$

PROPOSITION H.—THEOREM.

If a ratio compounded of ſeveral ratios be the ſame with a ratio compounded of any other ratios, and if one of the firſt ratios, or a ratio compounded of any of the firſt, be the ſame with one of the laſt ratios, or with the ratio compounded of any of the laſt, then the ratio compounded of the remaining ratios of the firſt, or the remaining ratio of the firſt, if but one remain, is the ſame with the ratio compounded of thoſe remaining of the laſt, or with the remaining ratio of the laſt.

$$\text{Hyp.}\left\{\begin{array}{l}A,B,C,D,E,F\\ G,H,K,L,M\end{array}\right\}\left\{\begin{array}{l}A:F=G:M\\ A:D=G:K\end{array}\right\}\begin{array}{c}\text{Then}\\ D:F=K:M.\end{array}$$

$$\left\{\begin{array}{l}
\left.\begin{array}{l} A:F=\overline{A:B}+\overline{B:C}+\overline{C:D}+\overline{D:E}+\overline{E:F} \\ G:M=\overline{G:H}+\overline{H:K}+\overline{K:L}+\overline{L:M} \end{array}\right\} A:F=G:M,\ \text{Hyp.} \\[2ex]
\left.\begin{array}{l} A:D=\overline{A:B}+\overline{B:C}+\overline{C:D} \\ G:K=\overline{G:K}+\overline{H:K} \end{array}\right] A:D=G:K,\text{Hyp.} \mid D:A=K:G,\text{Pr.B.} \\[2ex]
\hline
\left.\begin{array}{l} D:F=\overline{D:E}+\overline{E:F} \\ K:M=\overline{K:L}+\overline{L:M} \end{array}\right\} \qquad \left\{\begin{array}{c} \because \text{Prop. XXII.} \\ D:F=K:M \end{array}\right\}
\end{array}\right.$$

PROPOSITION K.—THEOREM.

If there be any number of ratios, and any number of other ratios, ſuch that the ratio compounded of ratios which are the ſame with the firſt ratios, each to each, is the ſame with the ratio compounded of ratios which are the ſame, each to each, with the laſt ratios; and if one of the firſt ratios, or the ratio which is compounded of ratios which are the ſame with ſeveral of the firſt ratios, each to each, be the ſame with one of the laſt ratios, or with the ratio compounded of ratios which are the ſame, each to each, with ſeveral of the laſt ratios; then the ratio compounded of ratios which are the ſame with the remaining ratios of the firſt, each to each, or the remaining ratio of the firſt, if but one remain, is the ſame with the ratio compounded of ratios which are the ſame with thoſe remaining of the laſt, each to each, or with the remaining ratio of the laſt.

	h, k, l	
A : B;	C : D; E, F	
G : H; K : L;	M : N; O, P; Q, R	S, T, V, X
e, f, g	m, n, o, p	Y, Z, a, b, c, d.

$$\left.\begin{array}{c} A:B,\ C:D,\ E:F \\ G:H,\ K:L,\ M:N,\ O:P,\ Q:R \end{array}\right\} \left.\begin{array}{l} A:B=S:T \\ C:D=T:V \\ E:F=V:X \end{array}\right. \left\{\begin{array}{l} G:H=y:z \\ K:L=z:a \\ M:N=a:b \\ O:P=b:c \\ Q:R=c:d \end{array}\right\}$$

$$\left.\begin{array}{l} S:x=y:d \\ A:B=e:g(=S:T) \\ h:l=\overline{C:D}+\overline{E:F} \\ m:p=\overline{M:N}+\overline{O:P}+\overline{Q:R} \end{array}\right\} \begin{array}{c} \text{Then} \\ h:l=m:p. \end{array}$$

$$\text{Def. X.}$$

$$S:X=\overline{S:T}+\overline{T:V}+\overline{V:X}=\overline{A:B}+\overline{C:D}+\overline{E:F},\ \text{Hyp.}$$

$$y:d=\overline{y:z}+\overline{z:a}+\overline{a:b}+\overline{b:c}+\overline{c:d}=\overline{G:H}+\overline{K:L}+\overline{M:N}+\overline{O:P}+\overline{Q:R},\text{hyp.}$$

$$e:g=\overline{e:f}+\overline{f:g}(=A:B=S:T,\ \text{Hyp.})=\overline{G:H}+\overline{K:L},\ \text{Hyp.}$$

$$h:l=\overline{h:k}+\overline{k:l}=\overline{C:D}+\overline{E:F},\ \text{Hyp.}$$

$$m:p=\overline{m:n}+\overline{n:o}+\overline{o:p}=\overline{M:N}+\overline{O:P}+\overline{Q:R},\text{Hyp.}=\overline{a:b}+\overline{b:c}+\overline{c:d},\text{Hyp.}$$

$$\text{Hyp.}\left\{\begin{array}{l} e:f(=G:H)=y:z \\ f:g(=K:L)=z:a \\ h:k(=C:D)=T:V \\ k:l(=E:F)=V:X \\ \quad - \quad - \quad - \\ m:p(=a:b+b:c+c:d)=a:d,\ \text{Prop. XXII.} \end{array}\right.$$

$$\left.\begin{array}{l} e:f=y:z \\ f:g=z:a \end{array}\right\} \because \text{Prop. XXII.}\ e:g=y:a \qquad \left\{\begin{array}{l} A:B(=S:T)=e:g,\ \text{Hyp.} \\ \because S:T=y:a,\ \text{Prop. XI.} \end{array}\right.$$

$$\left.\begin{array}{l} h:k=T:V \\ k:l=V:X \end{array}\right\} \because \text{Prop. XXII.}\ h:l=T:X \qquad \left\{\begin{array}{l} T:S=a:y,\ \text{Prop. B.} \\ S:X=y:d,\ \text{Hyp.} \\ \because T:X=a:d,\ \text{Pr. XXII.} \end{array}\right.$$

$$\left\{\begin{array}{l} h:l=T:X \\ T:X=a:d \\ a:d=m:p \end{array}\right\} \because \text{Prop. XI.}\ h:l=m:p.$$

FINIS.